UN TEOREMA QUE PERMITE OBTENER ECUACIONES DE OVOLUCION DETERMINISTAS A PARTIR DE ESTOCASTICAS DISCRETAS.

A THEOREM THAT ALLOWS TO FIND DETERMINISTIC EVOLUTION EQUATIONS FROM A SET OF DISCRETE STOCHASTIC EVOLUTION EQUATIONS.

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It is proved that it is possible to obtain continuum deterministic evolution equations from a set of discrete and stochastics rules after an average over realizations on the dynamical variables. Examples are given.

Keywords: Evolution equations; Stochastic procceses.

Se demuestra que es posible obtener las ecuaciones de evolución deterministas continuas a partir de un conjunto de reglas de evolución discretas y estocásticas después de realizar un promedio sobre realizaciones de las variables dinámicas. Se dan ejemplos de aplicación.

Palabras claves: Ecuaciones de evolución; Procesos estocásticos.

I. INTRODUCTION

Deterministic evolution equations that evolve Markovianly as well as non-Markovianly was the matter of investigations over decades and an illustrative list, including both type of equations, can be found in the works given in [1-22]. Discrete stochastic evolution equations were initially studied in [23], where three classes of evolution equations with weights and dynamical variables that are real numbers were studied. Subsequently, the extension to weights and dynamical variables that are complex numbers were studied in [24]. Additionally, non-Markovian discrete evolution equations [25], mix of Markovian and non-Markovian discrete evolution equations [26] and the extension to dynamical variables that are components of tensors [27], were subsequently studied. Deterministic discrete evolution equations were extensively studied and initiated by T. Regge [28] and more recently efforts in this direction can be found in [29].

In this paper, it will be extended the previous results to the most general version of the theorem that allows to obtain continuum deterministic evolution equations from a set of discrete stochastic evolution equations. The dynamical variables considered are, as in [27], tensors of any rank that can be in general complex numbers and the object of study are functions of discrete stochastic evolution equations like the Lagrangian and consequently perhaps the widest variety of problems can be studied with the present approach. This is the fifth extension of a theorem that allow to obtain the Euler equation for the Lagrangian and constitute a version that include the other four companion papers [24-27] as special cases.

The paper is organized as follows. In Section 2 an introduction of the evolution rules corresponding to models with an updating of the dynamical variables that depends on the values of functions of the dynamical variables at an arbitrary number of previous time steps and with subsets that are of different type is considered. In Section 3 a theorem allowing to connect two sets of stochastic evolution equations with another two sets that contain deterministic weights is proved. This connection is proved for the case of non-Markovian discrete stochastic evolution equations for sets of different type of dynamical variables. The Markovian case can be obtained as a special case with updating that depends only on the first previous time step. In Section 4 the general procedure will be applied to obtain the deterministic differential equation for two sets of equations. In the first set, studied in subsection 4.1, the dynamical variables with rules that are functions of the Hamiltonian and in the second set, studied in subsection 4.2, the derivation of the Euler equations for the Lagrangian is considered. In section 5 a discussion and the conclusions will be considered.

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II. STOCHASTIC EVOLUTION UPDATING FOR A SET OF FUNCTIONS OF COMPLEX DYNAMICAL VARIABLES THAT ARE COMPONENTS OF TENSORS: BASIC DEFINITIONS

A four-dimensional lattice Λ consisting of a set of points $\{\mathbf{x}\}$, with periodic boundary conditions in an interval $[-L_{0i}/2, +L_{0i}/2]$ for $i = 1, ..., 4$ (L_{0i} being finite or infinite) will be considered and a set of complex dynamical variables $\{q^{(r_0)r_1...r_{\alpha_{s_0}}}_{(s_0)s_1...s_{\beta_{s_0}}}(t, \mathbf{x})\}$ will be used for describing the value of each dynamical variable of type s_0 in a realization r_0 that are components of a tensor α_{s_0} time contravariant and β_{s_0} times covariant of order $\alpha_{s_0} + \beta_{s_0}$ at coordinate $\mathbf{x} = x_1, x_2, x_3, x_4$ and at evolution parameter t. The separation between sites or lattice constant is a_1, a_2, a_3, a_4 and the separation between two successive updates is a_0. The length of the lattice corresponding to each coordinate is $L_i = a_i(2L_{0i}+1)$ and the number of lattice sites is $M = (2L_{01}+1)(2L_{02}+1)(2L_{03}+1)(2L_{04}+1)$. The evolution equation for a set of functions of dynamical variable can be expressed in the following general form

$$\begin{aligned}\mathcal{F}\left(\{q^{(r_0)r_1...r_{\alpha_{s_0}}}_{(s_0)s_1...s_{\beta_{s_0}}}(t+a_0, \mathbf{x})\}\right) &= \mathcal{F}\left(\{q^{(r_0)r_1...r_{\alpha_{s_0}}}_{(s_0)s_1...s_{\beta_{s_0}}}(t, \mathbf{x})\}\right) + \mathcal{G}^{(r_0)}_{(s_0)}(t+a_0, t, ..., t-l_k a_0, X, X_0, ..., X_{l_{0k}}, X_j, X_\xi),\\ &\forall s_0 \in \{1, ..., S\}, t \geq 0, \mathbf{x} \in \Lambda, \qquad (1)\end{aligned}$$

where $\mathcal{G}$ denote the set of rules that define the updating corresponding to a given model or approach and S is the number of different type of variables. For example in the Hamiltonian and Lagrangian approach $S = 2$: the generalized coordinates (one type of variables, say ($s_0 = 1$)) and momenta or velocities (the other type ($s_0 = 2$)), respectively, as shown in the examples in section 4 below. Also, X, X_0,...,$X_{l_{0k}}$ denote the set of complex dynamical variables $\{q^{(r_0)r_1...r_{\alpha_{s_0}}}_{(s_0)s_1...s_{\beta_{s_0}}}(t+a_0, \mathbf{x})\}$, $\{q^{(r_0)r_1...r_{\alpha_{s_0}}}_{(s_0)s_1...s_{\beta_{s_0}}}(t, \mathbf{x})\}$,..,$\{q^{(r_0)r_1...r_{\alpha_{s_0}}}_{(s_0)s_1...s_{\beta_{s_0}}}(t - l_{0k}a_0, \mathbf{x})\}$, respectively. The set of both discrete and continuous stochastic variables that confer stochasticity to the evolution equations are $X_j = \{j\}$ and $X_\xi = \{\xi\}$, respectively. Note that both, $j = j^{(r_0)}(t)$ and $\xi = \xi^{(r_0)}(t)$, depend on the particular realization r_0 and on the evolution parameter t. Below, usually the dependence on t is neglected and in j also the dependence on r_0, in order to save printing. The sets of dynamical variables depends on the particular realization r_0, the time $t + a_0$, and previous time $t, ..., t - l_{0k}a_0$. The number of previous time is $k + 2$ and the set is $\{l_{0\alpha}\} = \{-1, 0, ..., k\}$, for any $-1 \geq \alpha \geq k$. The stochastic variables are chosen in such a way that all of them are statistically independent and a factorization of each product that contain stochastic variables is then possible. A particular case of Eq,(1) is the evolution of a set of dynamical variables, as those given in previous companion papers, with a general expression of the form

$$\begin{aligned}q^{(r_0)r_1...r_{\alpha_{s_0}}}_{(s_0)s_1...s_{\beta_{s_0}}}(t+a_0, \mathbf{x}) &= q^{(r_0)r_1...r_{\alpha_{s_0}}}_{(s_0)s_1...s_{\beta_{s_0}}}(t, \mathbf{x}) + G^{(r_0)}_{(s_0)}(t, ..., t-l_k a_0, X_0, ..., X_{l_{0k}}, X_j, X_\xi),\\ &\forall s_0 \in \{1, ..., S\}, t \geq 0, \mathbf{x} \in \Lambda, \qquad (2)\end{aligned}$$

where G is a set of rules that define the evolution of a given model.

Let us assume, for the sake of simplicity, that the set of S equations is separated in subsets containing S_1 and S_2 equations such that $S = S_1 + S_2$. Moreover, let us assume that the set of updating rules (also for the sake of simplicity) are of the following particular form

$$\begin{aligned}\mathcal{G} &= \sum_{\{s_{01},\mathbf{l}_1\}} w^{(r_0)}_{s_{01},\mathbf{l}_1} \quad \mathcal{F}\left(\{q^{(r_0)\mathbf{r}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}}(t - l_{01}a_0, \mathbf{x}_{11} + \Delta\mathbf{x}_{11}) + l_{q_{s_0},s_1}a_{q_{s0}}\}\right)\\ &\forall s_{01} \in \{1, 2\}, t \geq 0, x_1, ..., x_4 \in \Lambda, \qquad (3)\end{aligned}$$

where the short hand notation $\mathbf{r}_{\alpha s_0} = r_1...r_{\alpha_{s_0}}$, $\mathbf{s}_{\beta s_0} = s_1...s_{\beta_{s_0}}$ for $s_0 = 1, 2$ and $\mathbf{x}_{11} + \Delta\mathbf{x}_{11} = x_1 + l_{11}a_1, x_2 + l_{21}a_2, x_3 + l_{31}a_3 + l_{41}a_4$ was used. Even when at first glance $\mathcal{G}$ seems to be some particular case, is enough general to include the evolution equations in the Hamiltonian and Lagrangian approach, which are the starting point in field theory, and further extensions are more or less straightforward. Note that also an additional discretization of the dynamical variables itself was used with the unit $a_{q_{s_0}}$, and $l_{q_{s_0},s_1}$ is some integer. In order to *derive* the deterministic equations for $\mathcal{F}$, Eq.(1) with the set of rules given in Eq.(3) will be used as the starting set of stochastic evolution equations. The short hand notation $\mathbf{l}_1 = l_{01}, l_{11}, l_{21}, l_{31}, l_{41}$ was also used in order to save printing. The stochastic weights and the dynamical variables, in Eq.(3), are labeled with an index r_0 emphasizing that the value depends on a specific realization. The stochastic weights can, in general, be a complex number with a real $w^{'(r_0)}_{s_{01},\mathbf{l}_1}$ and an imaginary part $w^{''(r_0)}_{s_{01},\mathbf{l}_1}$. In order to be more formal, an arbitrary weight can be denoted by $w^{(r_0)}_u$ where u is some set of indexes $u_1, ..., u_\rho$ non necessarily of the same type as in Eq.(3). A general expression of a weight as a product of Kronecker deltas and Heaviside's functions can be written as

$$
\begin{aligned}
w^{(r_0)}_u \;=\; & \prod_{\{k'\}} \delta_{i_{k'},j_{k'}} \left(\prod_{\{v'\}} \theta \left(P_{v'} - \xi^{(r_0)}_{v'} \right) \right) \theta \left(P^{'}_{c'} - \xi^{'(r_0)}_{c'} \right) \\
& + \; i \prod_{\{k''\}} \delta_{i_{k''},j_{k''}} \left(\prod_{\{v''\}} \theta \left(P_{v''} - \xi^{(r_0)}_{v''} \right) \right) \theta \left(P^{''}_{c''} - \xi^{''(r_0)}_{c''} \right) .
\end{aligned} \tag{4}
$$

where $\{k'\}$ and $\{v'\}$ are sets of indexes that are used to label discrete and continuous factors, respectively. These indexes correspond to the real part of the complex weight $w^{(r_0)}_u$. c' denote the index that connect the real part of the stochastic weight with the real part of the deterministic weight of some other approach. In the same way, $\{k''\}$, $\{v''\}$ and c'' denote the indexes corresponding to the imaginary part of $w^{(r_0)}_u$. The imaginary unit is i.

There are some key questions that allows the construction of deterministic evolution equations from an average over realizations of a stochastic evolution equation. First, the stochastic weights must be expressed as products of some delta- and theta-functions whose arguments contain discrete as well as continuous stochastic variables, respectively. The definition of these functions are: $\delta_{x,y}$ is equal to 1 if $x = y$ and 0 otherwise, and $\theta(x-y)$ is equal to 1 if $x - y \geq 0$ and 0 if $x - y < 0$, for any x and y. Second, all these stochastic variables (discrete and continuous) are statistically independent, allowing the factorization of the averages. Third, two of the theta-functions, corresponding to the real and imaginary part of the stochastic weights, contain in its argument the functions $P^{'}_{c'}$ and $P^{''}_{c''}$ that allows to connect the average over realizations of all the stochastic weights with the deterministic weights of any other deterministic approach (e.g. master equation, etc.). For the interpretation of these functions that define the weights see the first example in Section 4 of [24].

The above general definition of a generic stochastic weight allows to demonstrate the following theorem.

III. A THEOREM CONNECTING THE AVERAGE OVER REALIZATIONS OF THE STOCHASTIC WEIGHTS WITH THE DETERMINISTIC WEIGHTS

The proof of the theorem can be made in an almost verbatim way, with the appropriate changes in the notation, that the one made in [25]. For the sake of completeness it is reproduced the theorem and the proof below.

Theorem. *A set of deterministic evolution equations is obtained after an average over realizations of a set of stochastic evolution equations like those given in* Eq.(1) *with the set of rules given in* Eq.(3) *which posses stochastic coefficients of the general form given in* Eq.(4). *The connection with a set of deterministic evolution equation, obtained with other approach, is made after an appropriate election of the functions* $P^{'}_{c'}$ *and* $P^{''}_{c''}$.

Proof. The proof is obtained in two steps in a very simple way. First, using standard results of statistical mechanics (see the appendix of [24]), the general deterministic equations are obtained after average over realizations on both two sides of Eq.(1) with the rules given in Eq.(3), in the following general form

$$
\begin{aligned}
\mathcal{F}\left(\{q^{\mathbf{r}_{\alpha 1}}_{(1)\mathbf{s}_{\beta 1}}(t+a_0,\mathbf{x})\}\right) \;&=\; \mathcal{F}\left(\{q^{\mathbf{r}_{\alpha 1}}_{(1)\mathbf{s}_{\beta 1}}(t,\mathbf{x})\}\right) + \sum_{\{s_{01},\mathbf{l}_1\}} w_{s_{01},\mathbf{l}_1} \\
&\times\; \mathcal{F}\left(\{q^{\mathbf{r}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}}(t-l_{0\alpha}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11})+l_{q_{s_0},s_1}a_{q_{s_0}}\}\right), \\
&\forall\, s_{01}\in\{1,2\}, t\geq 0, x_1,...,x_4\in\Lambda,
\end{aligned} \tag{5}
$$

where $w_{s_{01},\mathbf{l}_1} = \overline{w^{(r_0)}_{s_{01},\mathbf{l}_1}}$ are the corresponding deterministic weights. Note that it was also used $\mathcal{F}\left(\{q^{\mathbf{r}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}}(t-l_{0\alpha}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11})\}\right) = \overline{\mathcal{F}\left(\{q^{(r_0)\mathbf{r}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}}(t-l_{0\alpha}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11})\}\right)} = \mathcal{F}\left(\{\overline{q^{(r_0)\mathbf{r}_{\alpha s_{01}}}_{(s_{01})\mathbf{s}_{\beta s_{01}}}(t-l_{0\alpha}a_0,\mathbf{x}_{11}+\Delta\mathbf{x}_{11})}\}\right)$ for $\alpha \geq -1$, which is the simplest closure that can be used and usually correspond to a (1,0)-closure. The deterministic weights can be written in the usual form $w_{s_{01},\mathbf{l}_1} = w^{'}_{s_{01},\mathbf{l}_1} + i\, w^{''}_{s_{01},\mathbf{l}_1}$. Note that the factorization of the averages over realization was used because it was assumed that the discrete and continuous stochastic variables in all w's are statistically independent. For a demonstration that the product of two functions of complex stochastic variables factorizes, see the appendix of [24].

Second, the last step needed to obtain the connection between the approaches is to make an average over realizations on both two sides of Eq.(4). The result is

$$
\begin{aligned}
\overline{w^{(r_0)}_u} \;&=\; \prod_{\{k'\}} \overline{\delta_{i_{k'},j_{k'}}} \left(\prod_{\{v'\}} \overline{\theta\left(P_{v'}-\xi^{(r)}_{v'}\right)}\right) \overline{\theta\left(P^{'}_{c'}-\xi^{'(r)}_{c'}\right)} + i \prod_{\{k''\}} \overline{\delta_{i_{k''},j_{k''}}} \left(\prod_{\{v''\}} \overline{\theta\left(P_{v''}-\xi^{(r)}_{v''}\right)}\right) \overline{\theta\left(P^{''}_{c''}-\xi^{''(r)}_{c''}\right)} \\
&=\; \prod_{\{k'\}} \frac{1}{M_{k'}} \left(\prod_{\{v'\}} P_{v'}\right) P^{'}_{c'} + i \prod_{\{k''\}} \frac{1}{M_{k''}} \left(\prod_{\{v''\}} P_{v''}\right) P^{''}_{c''},
\end{aligned} \tag{6}
$$

where $M_{k'}$ and $M_{k''}$ are the number of element of the k-th discrete set. Note that it was assumed that all the intervals of variation of all the continuous stochastic variables is $[0,1]$. If some of the intervals is different, the result of Eq.(56) in the appendix of [24] must be used. The connection with another approach is easily obtained. Equating the coefficients of the expressions of the weights $\left(\overline{w^{(r_0)}_u} = w_c\right)$, $P^{'}_{c'}$ and $P^{''}_{c''}$ can be found as

$$
P^{'}_{c'} = \frac{w^{'}_c}{\prod_{\{k'\}} \frac{1}{M_{k'}} \left(\prod_{\{v'\}} P_{v'}\right)}, \tag{7}
$$

$$
P^{''}_{c''} = \frac{w^{''}_c}{\prod_{\{k''\}} \frac{1}{M_{k''}} \left(\prod_{\{v''\}} P_{v''}\right)}, \tag{8}
$$

where $w^{'}_c$ and $w^{''}_c$ are the real and imaginary part of w_c, respectively. If the deterministic evolution equation is expressed as a partial differential equation like those given in the example in section 4, $P^{'}_{c'}$ and $P^{''}_{c''}$, in Eqs.(7,8), must be multiplied by a_0, when necessary, in order to recover the correct deterministic weights. These expressions allows to establish the complete equivalence with the deterministic weights corresponding to some other approach.

IV. ILLUSTRATIVE EXAMPLES

In this section two illustrative examples are studied in detail in order to show the basic steps necessary to obtain the usual evolution equation for the generalized coordinates and momenta for the Hamiltonian approach. The second example correspond to the evolution equations for the Lagrangian which are the well known Euler equations.

A. Stochastic evolution updating for the discrete Hamiltonian approach

One of the usual way of constructing the evolution equations in physics is the Hamiltonian approach consisting in providing two subsets of evolution equations. The first subset are the evolution equation for the dynamical variables of type 1 ($s_0 = 1$) or generalized coordinates $q = q_1, ..., q_i, ..., q_n$ and the second for dynamical variables of type 2 ($s_0 = 2$) or the generalized momenta $p = p_1, ..., p_i, ..., p_n$. The corresponding evolution equations are the well known canonical equations given by

$$\frac{dq_i}{dt} = \frac{\partial \mathcal{H}(q,p)}{\partial p_i} \qquad for \qquad i = 1, ..., n, \quad (9)$$

$$\frac{dp_i}{dt} = -\frac{\partial \mathcal{H}(q,p)}{\partial q_i} \qquad for \qquad i = 1, ..., n, \quad (10)$$

where $\mathcal{H}(q, p)$ is the well known Hamiltonian function. This set of deterministic evolution equations are the most similar to the usual stochastic evolution equations used in previous companion papers [23-25]. In order to obtain the canonical evolution equations from a set of stochastic evolution equation the following discrete stochastic evolution equations are necessary:

$$\begin{aligned} q^{(r_0)}_{(1)s_1}(t+a_0) &= q^{(r_0)}_{(1)s_1}(t) + w^{(r_0)}_0 \mathcal{H}(q^{(r_0)}_{(1)}(t), q^{(r_0)}_{(2)}(t)) + w^{(r_0)}_1 \mathcal{H}(q^{(r_0)}_{(1)}(t), q^{(r_0)}_{(2)s1}(t) + a_{q_2}), \quad \forall s_1 \in \{1, ..., S_1\}, t \geq 0, \\ q^{(r_0)}_{(2)s_1}(t+a_0) &= q^{(r_0)}_{(2)s_1}(t) + w^{(r_0)}_2 \mathcal{H}(q^{(r_0)}_{(1)}(t), q^{(r_0)}_{(2)}(t)) + w^{(r_0)}_3 \mathcal{H}(q^{(r_0)}_{(1)s1}(t) + a_{q_1}, q^{(r_0)}_{(2)}(t)), \quad \forall s_1 \in \{S_1+1, ..., S\}, t \geq 0, \end{aligned} \quad (11)$$

where the last two summand on the right-hand side of both two equations are the corresponding rules G that define the evolution of the model. Note that it was used the notation $q^{(r_0)}_{(k)}(t) = \{q^{(r_0)}_{(k)s_1}(t)\}$ and $q^{(r_0)}_{(k)s_1}(t) + a_{q_k} = \{q^{(r_0)}_{(k)1}(t), ..., q^{(r_0)}_{(k)s_1}(t) + a_{q_k}, ..., q^{(r_0)}_{(k)n}(t)\}$ for $k = 1, 2$, in order to save printing in the arguments of $\mathcal{H}$. Note that in both two equations, $S_1 = S - S_1 = n$, was used. Another feature that posses the equations is that the dynamical variables $q^{(r_0)}_{(1)s_1}(t)$ and $q^{(r_0)}_{(2)s_1}(t)$ depend only on the evolution parameter t and do not depend explicitly on the coordinates $\mathbf{x}$. The subindex s_1 numerate the components of the dynamical variables and, in order to provide the same equations than those given above using the Hamiltonian approach, must be equal to n in each equation. This subindex correspond to i in the above approach. In order to simplify the notation the subindex in the weights were reduced to one number because all subindex included in $\mathbf{l}$ do not exist due that the dynamical variables of both two types do not depend explicitly on $\mathbf{x}$. Another new feature in these equations is that also the dynamical variables itself were discretized with unit intervals a_{q_1} and a_{q_2} corresponding to the dynamical variables of the type (1) and (2) respectively. The weights are $w_0 = \theta(P_0 - \xi^{(r_0)}_0)$, $w_1 = \theta(P_1 - \xi^{(r_0)}_1)$, $w_2 = \theta(P_2 - \xi^{(r_0)}_2)$, and $w_3 = \theta(P_3 - \xi^{(r_0)}_3)$, where $\theta(u)$ for any u is the Heaveside's function which takes the value 0 for $u < 0$ and 1 for $u \geq 0$. In the argument of the theta functions P_k and $\xi^{(r_0)}_k$, for any k, are the connecting parameters and the random numbers taken from the interval [0,1], respectively. The next step necessary to obtain the set of deterministic evolution equations is to average over realizations on both two sides of Eq.(11) given

$$\begin{aligned} q_{(1)s_1}(t+a_0) &= q_{(1)s_1}(t) + w_0 \mathcal{H}(q_{(1)s_1}(t), q_{(2)s_1}(t)) + w_1 \mathcal{H}(q_{(1)s_1}(t), q_{(2)s_1}(t) + a_{q_2}), \quad \forall s_1 = 1, ..., n, t \geq 0, \\ q_{(2)s_1}(t+a_0) &= q_{(2)s_1}(t) + w_2 \mathcal{H}(q_{(1)s_1}(t), q_{(2)s_1}(t)) + w_3 \mathcal{H}(q_{(1)s_1}(t) + a_{q_1}, q_{(2)s_1}(t)), \quad \forall s_1 = 1, ..., n, t \geq 0, \end{aligned} \quad (12)$$

where $q_{(s_0)s_1}(t) = \overline{q^{(r_0)}_{(s_0)s_1}(t)}$, $\mathcal{H}(q_{(1)s_1}(t), q_{(2)s_1}(t)) = \mathcal{H}(\overline{q^{(r_0)}_{(1)s_1}(t)}, \overline{q^{(r_0)}_{(2)s_1}(t)}) = \overline{\mathcal{H}(q^{(r_0)}_{(1)s_1}(t), q^{(r_0)}_{(2)s_1}(t))}$ and, as proved in [24], $\overline{w^{(r_0)}_k} = w_k = P_k$ for any k, was used. Expanding both two equations in a Taylor series up to $O(a_0)$, $O(a_{q_1})$ and $O(a_{q_2})$ and arranging terms, the following equations are obtained

$$\begin{aligned} &(w_0 + w_1)\mathcal{H}(q,p) - a_0 \frac{\partial q_i}{\partial t} + w_1 a_{q_2} \frac{\partial \mathcal{H}(q,p)}{\partial p_i} + O(a_0) + O(a_{q_1}) + O(a_{q_2}) = 0, \quad \forall \quad s_1 = 1, ..., n, \\ &(w_2 + w_3)\mathcal{H}(q,p) - a_0 \frac{\partial p_i}{\partial t} + w_3 a_{q_1} \frac{\partial \mathcal{H}(q,p)}{\partial q_i} + O(a_0) + O(a_{q_1}) + O(a_{q_2}) = 0, \quad \forall \quad s_1 = 1, ..., n, \end{aligned} \quad (13)$$

where, in order to obtain the same notation than the one usually used in the Hamiltonian approach, it was used the shorthand notation $q_i = q_{(1)s_1}(t)$ and $p_i = q_{(2)s_1}(t)$. Also $q = \{q_{(1)s_1}(t)\}$ and $p = \{q_{(2)s_1}(t)\}$ was used. The last step necessary to find the same equations than those given in Eqs.(9,10), the coefficients in Eq.(13) must satisfy the following set of linear equations

$$\begin{aligned}(w_0 + w_1) &= 0,\\ (w_2 + w_3) &= 0,\\ w_1 a_{q_2} &= a_0,\\ w_3 a_{q_1} &= -a_0,\end{aligned} \tag{14}$$

whose solution is $w_1 = \frac{a_0}{a_{q_2}}$, $w_3 = \frac{-a_0}{a_{q_1}}$, $w_0 = \frac{-a_0}{a_{q_2}}$ and $w_2 = \frac{a_0}{a_{q_1}}$. The connecting parameters are easily obtained and the corresponding values are $P_1 = w_1 = \frac{a_0}{a_{q_2}}$, $P_3 = w_3 = \frac{-a_0}{a_{q_1}}$, $P_0 = w_0 = \frac{-a_0}{a_{q_2}}$ and $P_2 = w_2 = \frac{a_0}{a_{q_1}}$, allowing to obtain the same deterministic equations as Eqs.(9,10).

B. Stochastic evolution updating for the discrete Lagrangian approach

The other usual way to obtain the evolution equation for the Lagrangian $\mathcal{L}(q, \dot{q})$, which is a function of the n dynamical variables $q = q_1, ..., q_i, ..., q_n$ and the corresponding velocities $\dot{q} = \frac{dq_1}{dt}, ..., \frac{dq_i}{dt}, ..., \frac{dq_n}{dt}$, is using a variational method and the following deterministic evolution equations for the lagrangian is obtained

$$\frac{\partial \mathcal{L}(q,\dot{q})}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}_i} = 0 \qquad for \quad i = 1, ..., n, \tag{15}$$

known as Euler's equations.

In order to find a set of deterministic evolution equations like Euler's equations, from a set of discrete stochastic evolution equations, the following updating rules are necessary

$$\begin{aligned}\mathcal{L}^{(r_0)}(q_{(1)}(t+a_0), \dot{q}_{(1)}(t+a_0)) \;&=\; \mathcal{L}^{(r_0)}(q_{(1)}(t), \dot{q}_{(1)}(t)) + \mathcal{G}^{(r_0)}(t, q_{(1)}(t), \dot{q}_{(1)}(t), X_\xi)\\ & \quad for \qquad i = 1, ..., n\end{aligned} \tag{16}$$

where

$$\begin{aligned}\mathcal{G}^{(r_0)}(t, q_{(1)}(t), \dot{q}_{(1)}(t), X_\xi) \;&=\; w_{00}^{(r_0)}\mathcal{L}^{(r_0)}(q_{(1)}(t), \dot{q}_{(1)}(t)) + w_{01}^{(r_0)}\mathcal{L}^{(r_0)}(q_{(1)i}(t) + a_{q_1}, \dot{q}_{(1)}(t))\\ &+\; w_{02}^{(r_0)}\mathcal{L}^{(r_0)}(q_{(1)}(t), \dot{q}_{(1)i}(t) + a_{q_2})\\ &+\; w_{12}^{(r_0)}\mathcal{L}^{(r_0)}(q_{(1)}(t+a_0), \dot{q}_{(1)i}(t+a_0) + a_{q_2}).\end{aligned} \tag{17}$$

The index r_0 indicates that the evolution equations correspond to a particular realization and the shorthand notation

$$\begin{aligned}\mathcal{L}^{(r_0)}(q_{(1)}(t), \dot{q}_{(1)}(t)) &= \mathcal{L}(q_{(1)1}^{(r_0)}(t), ..., q_{(1)i}^{(r_0)}(t), ..., q_{(1)n}^{(r_0)}(t), \dot{q}_{(1)1}^{(r_0)}(t), ..., \dot{q}_{(1)i}^{(r_0)}(t), ..., \dot{q}_{(1)n}^{(r_0)}(t)),\\ \mathcal{L}^{(r_0)}(q_{(1)i}(t) + a_{q_1}, \dot{q}_{(1)}(t)) &= \mathcal{L}(q_{(1)1}^{(r_0)}(t), ..., q_{(1)i}^{(r_0)}(t) + a_{q_1}, ..., q_{(1)n}^{(r_0)}(t), \dot{q}_{(1)1}^{(r_0)}(t), ..., \dot{q}_{(1)i}^{(r_0)}(t), ..., \dot{q}_{(1)n}^{(r_0)}(t)),\\ \mathcal{L}^{(r_0)}(q_{(1)}(t), \dot{q}_{(2)i}(t)) &= \mathcal{L}(q_{(1)1}^{(r_0)}(t), ..., q_{(1)i}^{(r_0)}(t), ..., q_{(1)n}^{(r_0)}(t), \dot{q}_{(1)1}^{(r_0)}(t), ..., \dot{q}_{(1)i}^{(r_0)}(t) + a_{q_1}, ..., \dot{q}_{(1)n}^{(r_0)}(t)),\end{aligned} \tag{18}$$

was used in order to save printing. Also $\dot{q}_{(1)}(t) = q_{(2)}(t)$ was used in order to have a notation similar to the usual approach given above.The stochastic weights are $w_{uv}^{(r_0)} = \theta(P_{uv} - \xi_{uv}^{(r_0)})$ for any u and v. In the argument of the Heaveside's function P_{uv} and $\xi_{uv}^{(r_0)} = \xi_{uv}^{(r_0)}(t)$ are the connecting parameter and a random number taken from an interval $[0,1]$ at time t. X_ξ, in the argument of $\mathcal{G}^{(r_0)}$, designate the set of stochastic variables $\{\xi_{uv}^{(r_0)}\}$, for any u and v. After an average over realizations on both two sides of Eq(16), with the rules given in Eq,(17), the following discrete deterministic equation is obtained

$$\begin{aligned}
\mathcal{L}(q_{(1)}(t+a_0),\dot{q}_{(1)}(t+a_0)) &= \mathcal{L}(q_{(1)}(t),\dot{q}_{(1)}(t)) + w_{00}\mathcal{L}(q_{(1)}(t),\dot{q}_{(1)}(t)) \\
&+ w_{01}\mathcal{L}(q_{(1)i}(t)+a_{q_1},\dot{q}_{(1)}(t)) + w_{02}\mathcal{L}(q_{(1)}(t),\dot{q}_{(1)i}(t)+a_{q_2}) \\
&+ w_{12}\mathcal{L}(q_{(1)}(t+a_0),\dot{q}_{(1)i}(t+a_0)+a_{q_2}), \\
& \quad for \qquad i=1,...,n
\end{aligned} \tag{19}$$

where a (1,0)-closure $\overline{\mathcal{L}(q_{(1)}(t)},\overline{\dot{q}_{(1)}(t))} = \overline{\mathcal{L}^{(r_0)}(q_{(1)}(t),\dot{q}_{(1)}(t))} = \mathcal{L}(\overline{q_{(1)}^{(r_0)}(t)},\overline{\dot{q}_{(1)}^{(r_0)}(t)})$ and the factorization $\overline{w_{uv}^{(r_0)}\mathcal{L}^{(r_0)}(q_{(1)}(t),\dot{q}_{(1)}(t))} = \overline{w_{uv}^{(r_0)}}\ \overline{\mathcal{L}^{(r_0)}(q_{(1)}(t),\dot{q}_{(1)}(t))}$ was used. As was proved in the appendix of [24], $\overline{w_{uv}^{(r_0)}} = \overline{\theta(P_{uv}-\xi_{uv}^{(r_0)})} = P_{uv}$ for any u and v . Expanding both two sides in Eq.(18) in a Taylor series up to $O(a_0)$, $O(a_1)$ and $O(a_2)$ the following deterministic differential equation is obtained

$$\begin{aligned}
-a_{q_1}w_{00}\frac{\partial\mathcal{L}(q_{(1)},\dot{q}_{(1)})}{\partial q_{(1)i}} &+ (w_{00}-w_{01}-w_{02}+w_{12})\,\mathcal{L}(q_{(1)},\dot{q}_{(1)}) \\
&+ a_{q_2}\,(w_{12}-w_{02})\,\frac{\partial\mathcal{L}(q_{(1)},\dot{q}_{(1)})}{\partial\dot{q}_{(1)i}} + 2a_0\,(w_{12}-1)\,\frac{\partial\mathcal{L}(q_{(1)},\dot{q}_{(1)})}{\partial t} \\
&+ 2a_0a_2w_{12}\frac{d}{dt}\frac{\partial\mathcal{L}(q_{(1)},\dot{q}_{(1)})}{\partial\dot{q}_{(1)i}} + O(a_0^2)+O(a_1^2)+O(a_2^2)=0 \\
&\qquad for \qquad i=1,...,n.
\end{aligned} \tag{20}$$

In order to obtain the Euler's equations, the coefficients must satisfy the following set of linear equations

$$\begin{aligned}
-a_{q_1}w_{00}+2a_0a_{q_2}w_{12} &= 0, \\
w_{00}-w_{01}-w_{02}+w_{12} &= 0, \\
w_{12}-w_{02} &= 0, \\
w_{12}-1 &= 0,
\end{aligned} \tag{21}$$

whose solution is $w_{00} = (2a_0a_{q_2})/a_{q_1}, w_{01} = (2a_0a_{q_2})/a_{q_1}, w_{02} = 1$ and $w_{12} = 1$. It is easy to see that Eq.(19) becomes

$$\begin{aligned}
&- 2a_0a_{q_2}\left(\frac{\partial\mathcal{L}(q,\dot{q})}{\partial q_i} - \frac{d}{dt}\frac{\partial\mathcal{L}(q,\dot{q})}{\partial\dot{q}_i}\right) + O(a_0^2)+O(a_{q_1}^2) \\
&+ O(a_{q_2}^2) = 0,\ \ for\ \ i=1,...,n,
\end{aligned} \tag{22}$$

which are, except for the coefficient $-2a_0a_2$, the well known Euler's equations. Finally, the connecting parameters can be easily obtained as $P_{00} = w_{00} = (2a_0a_{q_2})/a_1, P_{01} = w_{01} = (2a_0a_{q_2})/a_{q_1}, P_{02} = w_{02} = 1$ and $P_{12} = w_{12} = 1$. It must be emphasized that it was used the convenient shorthand notation $\mathcal{L}(q_{(1)},\dot{q}_{(1)}) = \mathcal{L}(q,\dot{q}) = \mathcal{L}(q_1(t),...,q_i(t),...,q_n(t),\dot{q}_1(t),...,\dot{q}_i(t),...,\dot{q}_n(t))$ in order to save printing.

V. CONCLUSIONS AND OTHER POSSIBLE GENERALIZATIONS

The extension of the discrete stochastic evolution equations approach to the case where the "object" that evolves are functions of the dynamical variables that are components of tensors of arbitrary rank, like the Euler's equations for the Lagrangian, was studied. A special case, where the function of the dynamical variables is the dynamical variable itself is studied initially in subsection 4.1 where the canonical equations for the generalized coordinates and momenta are derived. The most general case was considered in subsection 4.2 and the Euler equations that allow to find the Lagrangian, which is a function of the dynamical variables, were obtained. Other special cases, but not less important, of this theorem can be: 1) the product of dynamical variables or correlations, 2) the Hamilton-Jacobi equations for the action $\mathcal{S}$ which is also a function of the dynamical variables, etc. As can be easily seen the theorem is more general than the examples mentioned above because the evolution equations could be also non-Markovian and the dynamical variables can be tensors of any rank.

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