

COMPARACION ENTRE LAS SOLUCIONES DEL LIPIKIN-NOGAMI
Y RICHARDSON CON ENERGIA COMPLEJA EN EL MODELO DEL LIPKIN

COMPARISON BETWEEN THE LIPKIN-NOGAMI AND RICHARDSON
SOLUTIONS WITH COMPLEX SINGLE PARTICLE ENERGY IN THE LIPKIN'S MODEL

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Recibido: 07/02/17; aceptado: 02/10/17

La interacción de pairing es de gran interés debido a que es una de las componentes principales de la interacción residual en sistemas de muchos cuerpos. Las aproximaciones de Bardeen-Cooper-Schrieffer (BCS) y Lipkin-Nogami (LN) dan soluciones aproximadas al Hamiltoniano de pairing. Por otro lado, el pairing constante, admite solución exacta, llamada solución de Richardson. Los núcleos alejados de la línea de estabilidad tienen importantes correlaciones con los estados de dispersión y por ende, los estados del continuo deben ser tenidos en cuenta en forma explícita en la descripción de tales sistemas. Una forma de incluir el continuo es a través de los estados de Gamow, estos es, soluciones de la ecuación de Schroedinger con energía compleja. En este trabajo comparamos las soluciones aproximadas de BCS y LN con la solución exacta en el modelo simétrico de Lipkin con energía compleja. Encontramos que la solución de LN resulta muy parecida a la exacta y que la extensión de la solución de la BCS al plano de energía complejo produce soluciones puramente imaginarias para el gap cuando $G < G_c$.

Palabras claves: Lipkin-Nogami, Richardson, energía compleja, apareamiento.

The pairing interaction is one of the most important contribution of the residual interaction and then, it is of major importance for the study of many-body systems. One can get solutions of the pairing Hamiltonian through the Bardeen-Cooper-Schrieffer (BCS) or the Lipkin-Nogami (LN) approximations but, the pairing Hamiltonian admit exact solution worked out by Richardson. Nuclei far away from the stability line have important correlations with the continuum part of the energy spectrum, due that the Fermi level is very close to the continuum threshold. The correlations with the continuum can be included in the many-body description through the complex energy states, called Gamow states. In this work we compare the approximates and exact solutions of the pairing Hamiltonian in real and complex-energy representations. In the application of this formulation to the symmetric Lipkin model, we found that the LN solution is in a good agreement with the exact one; besides, the extension of the BCS solution to the complex energy plane gives solution even for strength below the critical one, which is purely imaginary.

Keywords: Lipkin-Nogami, Richardson, complex energy, pairing.

I. INTRODUCTION

Many-body configurations in the continuum is an important issue to understand the properties of loosely bound systems, as for example, nuclei close to the drip line [1]. The complex energy poles of the scattering matrix correspond to complex energy eigenvalue of a single particle Hamiltonian with purely outgoing boundary condition. They are called Gamow states and they represent decay states in the continuum [2]. They have information of the structure of the real energy continuum spectrum: the real part of the complex pole gives the resonant energy while the reciprocal of its imaginary part is proportional to half-life of the unstable state [3].

The constant pairing interaction, although simple, is an important component of the particle-particle

interaction [4, 5]. The pairing Hamiltonian can be solved using the Bardeen-Cooper-Schrieffer (BCS) approximation [6]; but, for small number of particles, this approximation is not a satisfactory solution. An improve solution was given by Nogami [7, 8] using the technique developed by Lipkin in Ref. [9], now known as Lipkin-Nogami (LN) approximation. The solutions in the LN approximation including the continuum single particle density was worked out recently in Ref. [10]. But, the constant pairing has exact solution worked out by Richardson [11, 12]. The eigenfunctions of the BCS and LN solutions do not conserve the number of particles of the system, while the Richardson solution does.

A test system, which is non trivial but it is simple enough to be exactly solvable was given by Lipkin, Meshkov and Glick [13], now known as Lipkin model or symmetric model. It is used to test the validity of new formalisms and techniques as well as to illustrate more complicated models in many-body systems [14–17].

In this work we calculated the many-body energy in the BCS, LN and Richardson frameworks with complex energy in the Lipkin model. The Lipkin formalism is described in section II. In section III the BCS, LN and Richardson solutions are given. The application to the Lipkin model is presented in section IV. Finally, in the last section V we draw some conclusions.

II. LIPKIN FORMALISM

Let us assume that $|\Phi\rangle$ is an approximate solution of the unknown exact ground state of a many-body system which is described by a Hamiltonian H . The wave function (w.f.) $|\Phi\rangle$ describes nicely some properties of the system but, at the same time, it violates some other property \hat{S} . For example, the BCS w.f. $|\Phi_{BCS}\rangle = \prod_{k>0}(1 + \phi_k)a_k^\dagger a_k^\dagger|0\rangle$ [6] describes nicely the pairing property of the many-body system but it is not an eigenfunction of the particle number operator \hat{N} . We may expand $|\Phi\rangle$ in a basis $|\phi_S\rangle$ of eigenfunctions of \hat{S} , i.e., $|\Phi\rangle = \sum_S c_S |\phi_S\rangle$, with $\langle\Phi|\Phi\rangle = 1$ ($\sum_S c_S^2 = 1$ and c_S real numbers.) In our example $|\Phi_{BCS}\rangle = \sum_{N=0}^{N_{max}} c_N |\phi_N\rangle$ where N is even, N_{max} is the maximum particle number allowed by the representation and $\hat{N}|\phi_N\rangle = N|\phi_N\rangle$.

An approximation of the ground state energy would be $\langle\Phi|H|\Phi\rangle$, but since $|\Phi\rangle$ does not conserve the property \hat{S} , we hope that $\langle\phi_S|H|\phi_S\rangle$ will be a better approximation for some specific value of S of the observable \hat{S} . The trick consist to use the approximate w.f. $|\Phi\rangle$ (called model w.f.), which we assume it is easier to handle, to obtain $\langle\phi_S|H|\phi_S\rangle$, together with a model Hamiltonian \mathcal{H} (to be build) for which our model w.f. is an eigenvector [9].

We defined our model Hamiltonian as

$$\mathcal{H} = H - f(\hat{S}), \quad (1)$$

with $f(\hat{S})|\phi_S\rangle = f(S)|\phi_S\rangle$, then

$$\langle\Phi|\mathcal{H}|\Phi\rangle = \langle\phi_S|H - f(S)|\phi_S\rangle \quad (2)$$

If $f(\hat{S})$ is chosen in such a way that $|\phi_S\rangle$ are all degenerate eigenfunctions of \mathcal{H} , i.e. $\mathcal{H}|\phi_S\rangle = \mathcal{E}|\phi_S\rangle$ then

$$\langle\Phi|\mathcal{H}|\Phi\rangle = \mathcal{E}. \quad (3)$$

By combining Eqs. (2) and (3) we get,

$$\mathcal{E} = \langle\phi_S|H - f(S)|\phi_S\rangle, \quad (4)$$

notice that \mathcal{E} is not the eigenvalue of our system but,

$$\langle\phi_S|H|\phi_S\rangle = \mathcal{E} + f(S), \quad (5)$$

then, the above discussion assume that $\mathcal{E} = \langle\Phi|\mathcal{H}|\Phi\rangle$ is easier to calculate than $\langle\phi_S|H|\phi_S\rangle$, i.e., it is easier to solve the eigenvalue problem for the model Hamiltonian $\mathcal{H}|\Phi\rangle$ than the original one $H|\phi_S\rangle$.

Probably, the only exactly known $f(\hat{S})$ is the momentum operator, in all the other cases this function has to be approximated by a Taylor's series, with the hope that a few terms will be enough to reproduce the truly ground state energy. So, let us assume that

$$f(\hat{S}) = f_1\hat{S} + f_2\hat{S}^2 + \dots \quad (6)$$

In the case that one truncates the series, the condition that $\langle\phi_S|\mathcal{H}|\phi_S\rangle$ be degenerated for all S is not fulfill. In such a case one must complement the problem with some other subsidiary condition.

If \hat{S} represents the particle number operator \hat{N} , the simplest approximation of $f(\hat{N})$ is when we keep the first term of the series, $f(\hat{N}) = \lambda\hat{N}$, then $\mathcal{H} = H - \lambda\hat{N}$, i.e. the BCS approximation. Then $\langle\Phi_{BCS}|\mathcal{H}|\Phi_{BCS}\rangle = \mathcal{E}_{BCS}$ and the subsidiary condition is that the mean value of the particle number operator is fixed, hence $\langle\phi_N|H|\phi_N\rangle = \mathcal{E}_{BCS} + \lambda N$. The condition $\langle\Phi_{BCS}|\hat{N}|\Phi_{BCS}\rangle = N$ determines the value of the parameter λ .

The equation

$$\langle\phi_S|H|\phi_S\rangle = \mathcal{E} + f(S) = \langle\Phi|\mathcal{H}|\Phi\rangle + f(S) \quad (7)$$

can be rearranged to be written,

$$\begin{aligned} \langle\phi_S|H|\phi_S\rangle &= \langle\Phi|H|\Phi\rangle + f_1(S - \langle\Phi|\hat{S}|\Phi\rangle) \\ &+ f_2(S^2 - \langle\Phi|\hat{S}^2|\Phi\rangle) + \dots \end{aligned} \quad (8)$$

where each term can be interpreted as a correction term. This way of writing the mean value $\langle\phi_S|H|\phi_S\rangle$ is a bit tricky. For example, in the BCS example

$$\begin{aligned} \langle\phi_N|H|\phi_N\rangle &= \langle\Phi_{BCS}|H|\Phi_{BCS}\rangle \\ &+ \lambda(N - \langle\Phi_{BCS}|\hat{N}|\Phi_{BCS}\rangle) \end{aligned} \quad (9)$$

and due the subsidiary condition $\langle\Phi_{BCS}|\hat{N}|\Phi_{BCS}\rangle = N$ the correction would be zero. The point we must remember is that we built the model Hamiltonian because it was easy to manipulate with our model wave function, i.e. we don't solve the eigenvalue problem $\langle\Phi|H|\Phi\rangle$, instead we solve the eigenvalue problem $\langle\Phi|\mathcal{H}|\Phi\rangle$. In this way we never face terms of the form $f_i(S^i - \langle\Phi|\hat{S}^i|\Phi\rangle)$ which could be zero. In our BCS example it means that we don't solve $\langle\Phi_{BCS}|H|\Phi_{BCS}\rangle$ but $\langle\Phi_{BCS}|H - \lambda\hat{N}|\Phi_{BCS}\rangle$.

The next step is to find a systematic way to obtain the parameters f_i for $i \geq 1$ which does not involve the states $|\phi_S\rangle$ but instead involves the model w.f. $|\Phi\rangle$. The model w.f. and the model Hamiltonian satisfies the relation

$$\langle\Phi|\mathcal{H}g(\hat{S})|\Phi\rangle = \langle\Phi|\mathcal{H}|\Phi\rangle \langle\Phi|g(\hat{S})|\Phi\rangle \quad (10)$$

for any function $g(\hat{S})$. We can choose a set of functions $g_i(\hat{S}) = \hat{S}^i$ with $i = 1, 2, \dots$ in order to evaluate the coefficients f_i . Then, a self-consistency conditions (independent of $|\phi_S\rangle$) is obtained by rearranged Eq. (10) with $g(\hat{S})$ replaced by \hat{S}^i ,

$$\langle\Phi|\mathcal{H}(\hat{S}^i - \langle\Phi|\hat{S}^i|\Phi\rangle)|\Phi\rangle = 0 \quad (11)$$

The application of this condition to our BCS example would give,

$$\langle \Phi_{BCS} | \mathcal{H} (\hat{N} - \langle \Phi_{BCS} | \hat{N} | \Phi_{BCS} \rangle) | \Phi_{BCS} \rangle = 0 \quad (12)$$

and then

$$\langle \Phi_{BCS} | \mathcal{H} \hat{N} | \Phi_{BCS} \rangle - \mathcal{E}_{BCS} \langle \Phi_{BCS} | \hat{N} | \Phi_{BCS} \rangle = 0 \quad (13)$$

By inserting $|\Phi_{BCS}\rangle\langle\Phi_{BCS}|$ between $\mathcal{H}\hat{N}$ and using the condition $\mathcal{H}_{20} = 0$ [18, 19] we get

$$\mathcal{E}_{BCS} \langle \Phi_{BCS} | \hat{N} | \Phi_{BCS} \rangle - \mathcal{E}_{BCS} N = 0 \quad (14)$$

which gives the standard condition used in BCS, $\langle \Phi_{BCS} | \hat{N} | \Phi_{BCS} \rangle = N$.

III. MODEL SOLUTIONS

The constant pairing Hamiltonian reads,

$$H = H_{sp} + V, \quad (15)$$

where

$$\begin{aligned} H_{sp} &= \sum_j \epsilon_j \hat{n}_j & \hat{n}_j &= \sum_m a_{jm}^\dagger a_{jm} \\ V &= -g P^\dagger P & P^\dagger &= \sum_{jm>0} a_{jm}^\dagger a_{j\bar{m}}^\dagger \end{aligned} \quad (16)$$

with $a_{j\bar{m}}^\dagger \equiv (-)^{j-m} a_{j,-m}^\dagger$, and g the strength of the interaction. The particle number operator is $\hat{N} = \sum_j \hat{n}_j$

A. Non conserving particle number solutions

In this section we will apply the Lipkin method [9] of section II to obtain approximate solutions of the pairing Hamiltonian (15).

1. BCS solution

The Taylor's expansion Eq. (6) in the particle number operator \hat{N} up to first order defines the usual BCS model Hamiltonian

$$\mathcal{H}_{BCS} = H - \lambda \hat{N}, \quad (17)$$

while the model w.f. is defined as [20]

$$|\Phi_{BCS}\rangle = \prod_{m>0} (u_j + v_j a_j^\dagger a_{j\bar{m}}^\dagger) |0\rangle \quad (18)$$

with the coefficients u_j and v_j satisfying $u_j^2 + v_j^2 = 1$. The ground state energy in this approximation is

$$E_{BCS} = \langle \Phi_{BCS} | \mathcal{H}_{BCS} | \Phi_{BCS} \rangle + \lambda N \quad (19)$$

$$= \sum_{jm} (\epsilon_j - \frac{g}{2} v_j^2) v_j^2 - \frac{\Delta^2}{g} \quad (20)$$

with

$$\begin{aligned} v_j^2 &= \frac{1}{2} \left(1 - \frac{\epsilon_j}{E_j} \right) \\ E_j &= \sqrt{\epsilon_j^2 + \Delta^2} \\ e_j &= \epsilon_j - \lambda - g v_j^2 \end{aligned} \quad (21)$$

The gap parameter Δ and the Fermi level λ are obtained by solving the following system of equations

$$\frac{4}{g} = \sum_j \frac{(2j+1)}{E_j} \quad (22)$$

$$N = \sum_j (2j+1) v_j^2 \quad (23)$$

2. Lipkin-Nogami solution

The model Hamiltonian we obtain by taking the Taylor's expansion Eq. (6) in the particle number operator \hat{N} up to the second order defines the Lipkin-Nogami (LN) model Hamiltonian [7]

$$\mathcal{H}_{LN} = H - \lambda_1 \hat{N} - \lambda_2 \hat{N}^2, \quad (24)$$

with H as Eq. (15). The model w.f. $|\Psi_{LN}\rangle$ is like Eq. (18) but with different coefficients u_j and v_j . They are determined in terms of the parameters Δ , λ_1 and λ_2 by solving the following system of three equations

$$\frac{4}{g} = \sum_j \frac{(2j+1)}{E_j} \quad (25)$$

$$N = \sum_j (2j+1) v_j^2 \quad (26)$$

$$\frac{4\lambda_2}{g} = \frac{(\sum_{jm} u_j^3 v_j)(\sum_{jm} u_j v_j^3) - 2 \sum_{jm} (u_j v_j)^4}{(\sum_{jm} (u_j v_j)^2)^2 - 2 \sum_{jm} (u_j v_j)^4} \quad (27)$$

with $\sum_m = 2j+1$ and

$$\begin{aligned} v_j^2 &= \frac{1}{2} \left(1 - \frac{\epsilon_j}{E_j} \right) \\ E_j &= \sqrt{\epsilon_j^2 + \Delta^2} \\ e_j &= \epsilon_j - \lambda + (4\lambda_2 - g) v_j^2 \\ \lambda &= \lambda_1 + 2\lambda_2(N+1) \end{aligned}$$

The ground state energy is

$$\begin{aligned} E_{LN} &= \langle \Phi_{LN} | \mathcal{H}_{LN} | \Phi_{LN} \rangle + \lambda_1 N + \lambda_2 N^2 \\ &= \sum_{jm} (\epsilon_j - \frac{g}{2} v_j^2) v_j^2 - \frac{\Delta^2}{g} - \lambda_2 \sum_{jm} 2u_j^2 v_j^2 \end{aligned} \quad (28)$$

B. Conserving particle number solution

The conserving particle number solution for a system of even N fermions is given in terms $N_{pairs} = N/2$

parameter E_n called pair energies. These parameters are obtained by solving a system of N_{pairs} equations, called Richardson's equations [11, 12]

$$\frac{1}{g} - \frac{1}{2} \sum_j \frac{2j+1}{2\epsilon_j - E_n} + 2 \sum_{n'=1, n' \neq n}^{N_{pairs}} \frac{1}{E_{n'} - E_n} = 0 \quad (29)$$

The many-body ground state energy E_{Rich} is defined taken the lowest N_{pairs} pair-energy E_n

$$E_{Rich} = \sum_n^{N_{pairs}} E_n \quad (30)$$

IV. APPLICATION: LIPKIN MODEL

The Lipkin model, also called symmetric model, consist of two equally degenerate levels with energies ϵ_u and ϵ_d at half filling. The notation of the previous section reduces to $\sum_{j,m} = \sum_{u,d} \sum_{m=-\Omega}^{\Omega}$. Hence, the degeneracy for each level is 2Ω and $N = 2\Omega$. Let us introduce the parameter $\epsilon > 0$

$$\epsilon = \epsilon_u - \epsilon_d, \quad (31)$$

which defines the energy separation between the two levels. The following values are used for the applications:

$$\begin{aligned} N &= 10 \\ \epsilon_d &= -0.5 \text{ MeV} \\ \epsilon_u &= (0.5 - i\gamma) \text{ MeV} \end{aligned}$$

By using the relations of the previous section we found the following algebraic solutions for the BCS and LN approximations:

a. *BCS solution:*

$$\begin{aligned} \lambda &= \frac{\epsilon_u + \epsilon_d}{2} - \frac{g}{2} \\ E &= E_u = E_d = g \Omega \\ \Delta &= g \Omega \sqrt{1 - \frac{\epsilon^2}{(2g\Omega - g)^2}} \end{aligned}$$

with

$$g > g_c \equiv \frac{\epsilon}{2\Omega - 1} \quad (32)$$

The ground state energy E_{BCS} Eq. (19), relative to the non-interacting system is

$$E_{BCS} - 2\Omega\epsilon_d = \Omega \left(\epsilon - \frac{g}{2} \right) - \frac{\Delta^2}{g} - \frac{\Omega e}{E} \left(\epsilon + \frac{g}{2} \chi \right) \quad (33)$$

where

$$\chi = \frac{\epsilon}{2\Omega g - g} \quad (34)$$

b. *LN solution:* The Lipkin-Nogami solution can also be obtained analytically,

$$\begin{aligned} \lambda &= \frac{\epsilon_u + \epsilon_d}{2} + \frac{\alpha}{2}; \quad \alpha = 4\lambda_2 - g > 0 \\ E &= E_u = E_d = g \Omega \\ \Delta &= g \Omega \sqrt{1 - \frac{\epsilon^2}{(2g\Omega + \alpha)^2}} \end{aligned}$$

with

$$g > g_c \equiv \frac{\epsilon - \alpha}{2\Omega} \quad (35)$$

Since $4\lambda_2 - g > 0$ we are interested in the positive α solution of the following cubic equation (see also Eq. (15) in Ref. [8])

$$\alpha(2\Omega - 1)[(2g\Omega + \alpha)^2 - \epsilon^2] - 2g\Omega\epsilon^2 = 0 \quad (36)$$

Figure 1 shows the three possible solutions of Eq. (36) for each value of the strength g in the range $[0, 0.5]$ MeV. As the strength goes to zero, the parameter α goes to zero and to $\pm\epsilon$ (with $\epsilon = 1$ MeV).

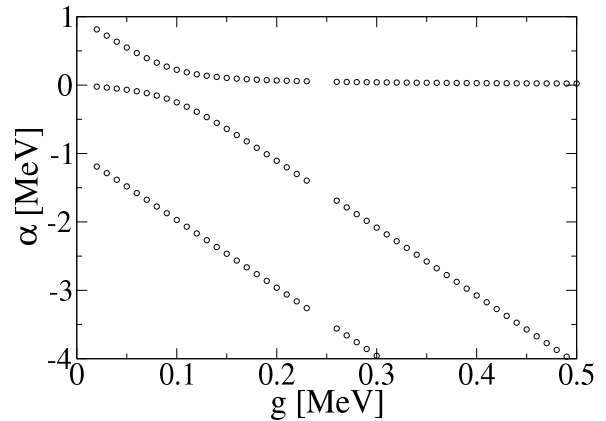


FIG. 1: Solutions of the cubic equation (36) $g \in [0, 0.5]$ MeV. Missing points in the figure is because no convergence was found.

The ground state energy E_{LN} Eq. (28) relative to the non-interacting Fermi energy $2\Omega\epsilon_d$ gives,

$$E_{LN} - 2\Omega\epsilon_d = \Omega \left(\epsilon - g - \frac{\alpha}{2} \right) - \frac{\Delta^2}{g} - \Omega\chi \left(\epsilon - \frac{\alpha}{2} \chi \right) \quad (37)$$

where

$$\chi = \frac{\epsilon}{2\Omega g + \alpha} \quad (38)$$

Equations (32) and (35) show that in order to find solution for the BCS and LN approximations, respectively, the strength g has to be greater than g_c . Figure 2 shows the value of g_c as a function of g for $\alpha > 0$. It is found that in order to have non trivial solution

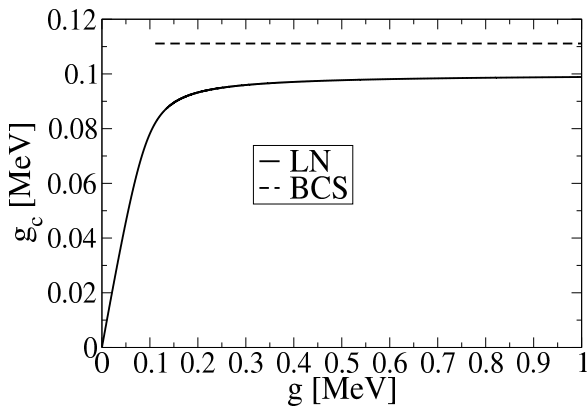


FIG. 2: g_c of Eqs. (32) and (35) (for $\alpha > 0$) versus g for the BCS and LN approximations, respectively.

in the BCS approximation g has to be bigger than a threshold value, while the LN approximation has no trivial solution for any value of the strength.

Figure 3 extends the comparison done in Fig. 1 of Ref. [8] between the BCS and LN gap parameter Δ as a function of the strength g for $\alpha > 0$ and $\gamma = 0$ to stronger strength. They compare well for strong correlation but they depart each other for small values of the strength. The figure also shows the nonphysical behavior of the pairing gap in the BCS approximation, i.e. $\Delta = 0$ for $g \lesssim 0.1$ MeV.

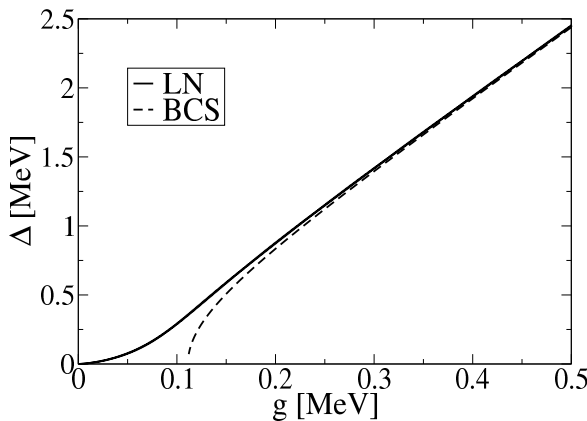


FIG. 3: Pairing gap versus g .

In Fig. 4 we show the gap parameter as complex magnitude in the BCS approximation for $\gamma = 0, 0.05,$ and 0.25 MeV. For complex energy the gap parameter is also complex but for $\gamma = 0$ the gap is purely real up to a minimum value and then it becomes purely imaginary, i.e., the trivial solution appears here as a complex solution with Δ purely imaginary; hence, in the complex plane the constant gap has no trivial solution for any value of g .

The ground state energy relative to the free system

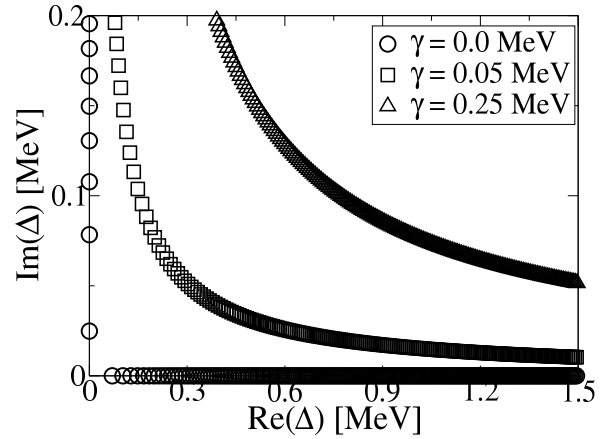


FIG. 4: Real and imaginary part of the pairing parameter calculated in the BCS approximation for different values of γ for $g < 0.32$ MeV.

of the non-conserving particles solutions are compared with the exact Richardson solution in Fig. 5 for $\gamma = 0$. We observe a very good agreement between the approximate LN and the exact (Richardson) solutions for all values of the pairing strength.

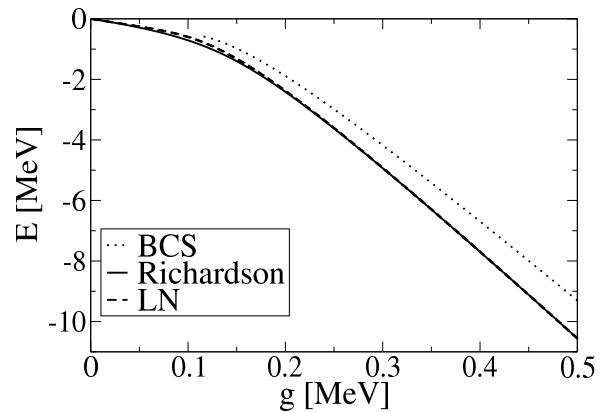


FIG. 5: Binding energy relative to the non-interacting Fermi sea versus g for $\gamma = 0$.

The ground state energy is calculated in the BCS and LN approximations for $\gamma = 0.25$ MeV and shown in Fig. 6. The energy of the BCS approximation diverges for values of the strength for which its gap is purely imaginary. While the imaginary part of the energy are similar in both approximations, the real one differs for the same value of the strength. Figure 6 also compares the real and imaginary parts of the energy with the exact Richardson solution. A good agreement with the LN approximation for all values of g can be observed.

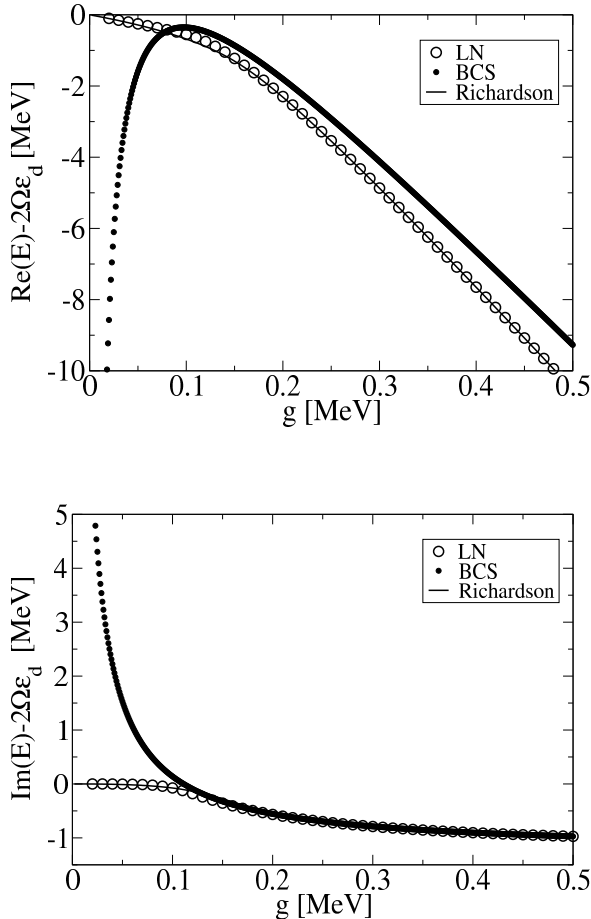


FIG. 6: Real (upper graph) and imaginary (lower graph) parts of the ground state energy for $\gamma = 0.25$ MeV as a function of the pairing strength g for the BCS, Lipkin-Nogami and Exact solutions.

V. DISCUSSION AND CONCLUSIONS

The Lipkin model with complex energy had been solved in the BCS and LN approximations and compared with the exact Richardson solution.

The extension of the pairing solution to the complex energy plane, shows that there is solution for any value of the strength, even in the BCS approximation, but this solution is completely nonphysical since the energy diverges.

It was found the LN approximation agreed very well with the exact Richardson solution for real and complex energy for any value of the strength.

A limitation of the exact Richardson solution is that it can be applied only to constant pairing interaction. But the good agreement with the LN solution seems to indicate that, for more general interactions, the LN method would be a well founded alternative, even in a complex energy representation.

VI. ACKNOWLEDGMENTS

This work has been funded by the National Council of Research (Consejo Nacional de Investigaciones Científicas y Técnicas - CONICET) by the grant PIP-625.

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