Exact solutions of the Navier-Stokes equation for steady parallel viscocapillary flows on an inclined plane

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Investigamos las soluciones de la ecuación de Navier-Stokes que describen flujos viscocapilares estacionarios paralelos sobre un plano inclinado. La forma de la superficie libre es dada por una fórmula analítica obtenida al resolver la ecuación que expresa el equilibrio estático bajo la acción de la gravedad y de la tensión superficial, independientemente del campo de velocidades y de cualquier hipótesis respecto de la reología del líquido. Luego, el campo de velocidades es obtenido resolviendo (en general, numéricamente) una ecuación de Poisson en el dominio definido por la sección transversal del flujo. Los contornos de isovelocidad son perpendiculares a la superficie libre. Se dan varias propiedades de las soluciones como funciones de los parámetros del problema. Se presentan dos soluciones analíticas especiales.

We investigate the solutions of the Navier-Stokes equations that describe the steady parallel viscocapillary flows down an inclined surface. The shape of the free surface is given by an analytic formula obtained by solving the equation that expresses the static equilibrium under the action of gravity and surface tension, independently of the velocity field and on any assumption concerning the rheology of the liquid. The velocity field is then obtained by solving (in general numerically) a Poisson equation in the domain defined by the cross section of the rivulet. The isovelocity contours are perpendicular to the free surface. Various properties of the solutions are given as functions of the parameters of the problem. Two special analytic solutions are presented.

I. INTRODUCTION

We investigate exact solutions of the Navier-Stokes equations that describe the steady flow of a rivulet down an incline. Flows over inclined solid surfaces under the action of gravity are ubiquitous in nature as well as in industrial processes. The theory of these currents is usually developed within the frame of the lubrication approximation. Flows on an horizontal plane have been studied theoretically and in the laboratory by several authors (see for example [1–4]). The equations for the same problem but on a general topography have been derived in [5].

Exact solutions of the Navier-Stokes equation for these flows can be of interest for several reasons. First, they represent fundamental fluid dynamic flows and owing to the uniform validity of exact solutions, the basic phenomena involved can be studied in detail, thus providing a valuable insight that can help in understanding related problems. Second, they allow to test the validity of approximations such as the lubrication theory. Finally the exact solutions serve as standards to check the accuracy of numerical simulations. These points have been stressed in [6], in which the reader can find an important list of references that complement the classical treatise [7] on exact solutions of the Navier-Stokes equation.

We investigate steady parallel flows whose free surface shape is determined by surface tension and gravity. Surface tension also provides the lateral confinement to a rivulet flowing down an inclined plane. In [8] exact solutions of the Navier-Stokes equation were obtained for viscocapillary flows in a rectangular inclined channel.

In Sec. II we derive the basic equations, and show that the shape of the free surface is determined by a static equilibrium condition, and does not depend on the velocity field. In Sec. III we obtain general analytical formulae for the free surface. In Sec. IV we present the equations that determine the velocity field, and in Sec. V we describe two special closed form solutions. In general, however, the velocity field must be obtained numerically; typical results are shown in Sec. VI. Section VII contains the concluding remarks.

II. BASIC EQUATIONS

We consider the steady flow of a rivulet running downslope on a plane whose inclination is α ($0 \leq \alpha \leq \pi/2$) (Fig. 1). The x, y coordinates lie in the plane (y is horizontal, and x increases downwards), and z is perpendicular to it. The rivulet extends from $-\infty < x < +\infty$ and from $-d/2 < y < d/2$, the velocity of the fluid is $\mathbf{u} = U(y,z)\hat{\mathbf{x}}$, and its free surface is given by $H \equiv H(y)$.

The continuity equation is clearly satisfied, and the Navier-Stokes equation reduces to

$$p_x = \mu(U_{yy} + U_{zz}) + \rho g \sin\alpha, \tag{1}$$
$$p_y = 0, \tag{2}$$
$$p_z = -\rho g \cos\alpha, \tag{3}$$

where p is the pressure, ρ is the density, g is the gravity,

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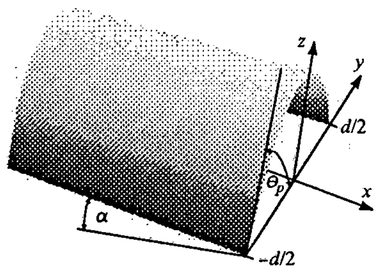

FIG. 1: Geometry of the problem.

μ is the viscosity, and we denote the derivatives of p and U with appropriate suffixes.

The boundary condition at the free surface is

$$\sigma \cdot \hat{\mathbf{n}} = \gamma C \hat{\mathbf{n}}. \tag{4}$$

Here $C = H''(1+H'^2)^{-3/2}$ is the curvature and $\hat{\mathbf{n}}$ is the normal of the free surface (the primes denote the derivatives of H with respect to y), γ is the surface tension, and the components of the stress tensor σ are

$$\sigma = \begin{bmatrix} -p & \mu U_y & \mu U_z \\ \mu U_y & -p & 0 \\ \mu U_z & 0 & -p \end{bmatrix}.$$

Equation (4) leads to the conditions

$$p = -\gamma C \quad \text{and} \quad U_z = H' U_y \quad \text{at } z = H. \tag{5}$$

Integrating (3) and using (5) we obtain $p = \rho g (H - z)\cos\alpha - \gamma C$, so that the pressure is hydrostatic. Using this result in (2) we obtain:

$$kH' = \left[\frac{H''}{(1+H'^2)^{3/2}}\right]', \tag{6}$$

where we have set $k = (\rho g/\gamma)\cos\alpha$ (k is the inverse of the square of the capillary length). Equation (6) must be solved subject to the boundary conditions

$$H(\pm d/2) = 0, \quad H'(-d/2) = -H'(d/2) = \tan\theta_p, \tag{7}$$

where θ_p is the static contact angle. Notice that $H(y)$ is determined by a static equilibrium condition, being independent of U, as the fluid motion is always parallel to the free surface. It can also be noticed that $H(y)$ is independent of the rheology, so that the present treatment can be extended to a non-Newtonian fluid.

The equation for U is

$$U_{yy} + U_{zz} = -\frac{\rho g \sin\alpha}{\mu}, \tag{8}$$

subject to the boundary conditions

$$U(y, 0) = 0, \quad U_z(y, H) - H' U_y(y, H) = 0. \tag{9}$$

If the rivulet is flowing over a non planar surface whose relief depends only on y and not on x, we obtain the same equations except that the boundary conditions (7) at the contact line and (9) at the bottom must be changed according to the relief of the supporting surface.

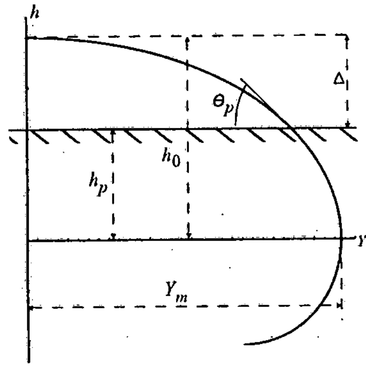

FIG. 2: The free surface.

III. THE SHAPE OF THE FREE SURFACE

The classical study of static menisci can be found in [9], and is amply discussed in [10]. For discussions of the physics of the static contact line see [11, 12]. We denote by Y, H the coordinates of a point of the free surface. For reason that will be apparent later it is convenient to employ a vertical coordinate z' whose origin is the plane where the slope of the solution of (6) takes the value $\theta = \pm\pi/2$ (see Fig. 2). Then $z' = h_p$ is the supporting plane ($z = 0$; $h_p > 0$ if $\theta_p < \pi/2$ or $h_p < 0$ if $\theta_p > \pi/2$) and the free surface is given by $h = H + h_p$.

We set $\Delta = H(0)$ and $h_0 = \Delta + h_p$ so that (6) now reads

$$kh' = \left[\frac{h''}{(1+h'^2)^{3/2}}\right]', \tag{10}$$

and (7) are converted into

$$h(-d/2) = h(d/2) = h_p, \quad h'(-d/2) = -h'(d/2) = \tan\theta_p.$$

Integrating (10) from 0 to Y, multiplying by h', and integrating again, we obtain

$$\frac{1}{2}k(h-h_0)^2 = 1 - \frac{1}{\sqrt{1+h'^2}} - (h-h_0)h''(0). \tag{11}$$

Calling Y_m the value of Y for which $h(Y_m) = 0$ and $h'(Y_m) = -\infty$ and using (11) we find

$$h''(0) = \frac{1}{2}kh_0 - \frac{1}{h_0}. \tag{12}$$

We observe that $h''(0)$ is the curvature of the free surface at the vertex, that is negative. On the other hand $|h''(0)|$ must be smaller than h_0^{-1} (that corresponds to the circular shape that is achieved when surface tension dominates). Then one finds that

$$0 \le \beta \equiv \frac{1}{2}kh_0^2 \le 1. \tag{13}$$

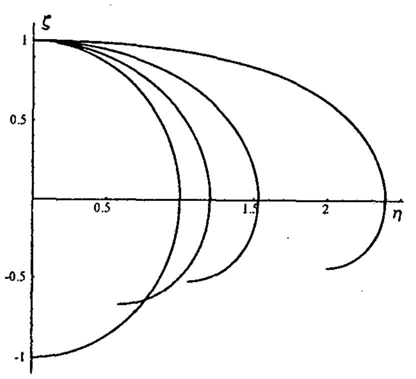

FIG. 3: The shape of the free surface given by (15). From left to right, the values of β are 0, 0.3, 0.6 and 0.9.

Substituting (12) in (11) we obtain

$$\frac{h}{h_0} + \frac{1}{2}kh(h_0 - h) = \frac{1}{\sqrt{1+h'^2}}. \qquad (14)$$

Introducing the dimensionless variables $\zeta = h/h_0$, $\eta = Y/h_0$, we can solve (14) for $d\eta/d\zeta$ to obtain

$$\frac{d\eta}{d\zeta} = \pm\frac{\zeta[1+\beta(1-\zeta)]}{\sqrt{1-\zeta^2[1+\beta(1-\zeta)]^2}}.$$

For $\beta < 1$, this equation can be integrated to find

$$\eta = \pm\left\{\frac{\sqrt{1-\zeta^2[1+\beta(1-\zeta)]^2}}{1-\beta\zeta} + \sqrt{\frac{2}{a\beta}}\left[F(\phi \mid a^2) - E(\phi \mid a^2)\right]\right\}. \qquad (15)$$

Here $F(\phi \mid a^2)$ and $E(\phi \mid a^2)$ are, respectively, the elliptic integrals of the first and second kind, and

$$\phi = \sqrt{\frac{\beta(1-\beta)}{a(1-\beta\zeta)}}, \quad a = \frac{\sqrt{1+\beta(6+\beta)}+1-\beta}{\sqrt{1+\beta(6+\beta)}-1+\beta}.$$

Notice that $a > 1$. In Fig. 3 we show $\eta(\zeta)$ for several values of β; it can be seen that the cross section of the rivulet tends to a circle as $\beta \to 0$, and becomes flatter and wider as $\beta \to 1$. In the limit $\beta = 1$ the width of the rivulet is infinite and one has

$$\eta = \pm\left\{\frac{d}{2} - \left[1 - \sqrt{1+h(2-h)} + \frac{1}{\sqrt{2}} \times \left(\operatorname{arctanh}\sqrt{\frac{1+h(2-h)}{2}} - \operatorname{arctanh}\frac{1}{\sqrt{2}}\right)\right]\right\}$$

with $d = \infty$. This solution is related to the cylindrical meniscus [9].

It is convenient to express the equation of the free surface using the slope $\theta = \arctan(d\zeta/d\eta)$ as a parameter. We obtain

$$\zeta = \frac{1}{2\beta}\left(1+\beta-\sqrt{(1+\beta)^2 - 4\beta\cos\theta}\right) \qquad (16)$$

$$\eta = \frac{1}{2\beta(1-\beta)}\left\{(1-\beta)^2 E\left(\frac{\theta}{2} \mid -\frac{8\beta}{(1-\beta)^2}\right) - (1+\beta)^2 F\left(\frac{\theta}{2} \mid -\frac{8\beta}{(1-\beta)^2}\right)\right\} \qquad (17)$$

Now we can appreciate the rationale of our change of the origin of the coordinates: by this device we obtained expressions of ζ and η that depend only on β, and not on θ_p. The contact angle only determines the position ζ_p of the supporting plane, given by $\zeta_p = \zeta(\theta_p)$. The width of the wetted stripe of the supporting plane is $d = 2\eta_p = 2\eta(\theta_p)$ and we can define the aspect ratio of the rivulet as $R = (1-\zeta_p)/d$.

Of course β depends on h_0, that is not directly observed in the actual profile. For this reason we need formulae that relate h_0 and β to some property of the visible shape of the rivulet, such as its maximum thickness Δ. Defining

$$\lambda = \frac{1}{2}k\Delta^2 = \frac{\rho g\cos\alpha}{2\gamma}\Delta^2$$

we find $h_0 = \delta\Delta$ with

$$\delta = \frac{1}{2\lambda}\left(\lambda - 1 + \cos\theta_p + \sqrt{(\lambda - 1 + \cos\theta_p)^2 + 4\lambda}\right),$$

$$\beta = 1 + \delta(\lambda - 1 + \cos\theta_p),$$

that are the desired formulae. Notice that the condition (13) is equivalent to the static equilibrium condition

$$\lambda = \frac{1}{2}k\Delta^2 \leq 1 - \cos\theta_p,$$

that sets the upper limit to the thickness of the rivulet.

IV. THE VELOCITY FIELD

We introduce $t = z/h_0$, $s = y/h_0$, $u = U/U_0$ where $U_0 = \mu^{-1}\rho h_0^2 g\sin\alpha$, so that (8) becomes the Poisson equation

$$u_{ss} + u_{tt} = -1, \qquad (18)$$

and the boundary conditions (9) read now

$$u(s,\zeta_p) = 0, \;\; u_t(s,\zeta) - \tfrac{d\zeta}{d\eta}u_s(s,\zeta) = 0. \qquad (19)$$

It can be verified that the last condition implies that the lines of equal velocity are perpendicular to the free surface. In general the problem (18-19) must be solved numerically. Before discussing the results, we show two special solutions that can be obtained in closed form.

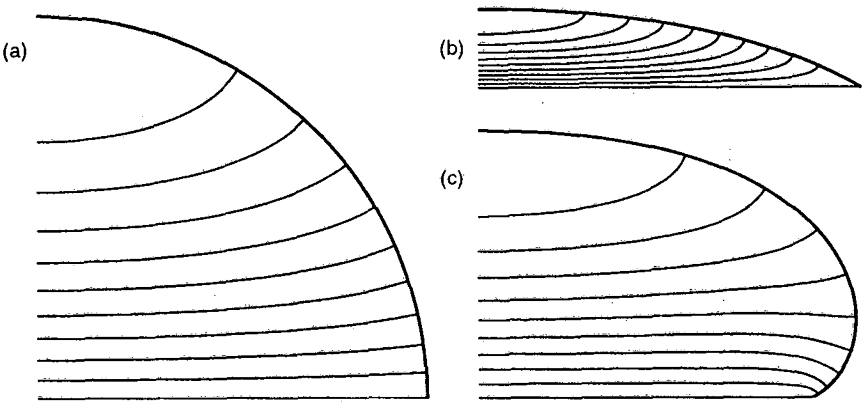

FIG. 4: Contour plots of: (a) the solution (26), (b) and (c) the numerical solutions of (18) and (19) for $\beta = 0.8$ and $\theta_p = \pi/6$ and $\theta_p = 5\pi/6$, respectively. The thick line is the free surface, and the thin lines are equally spaced isovelocity contours.

V. SPECIAL ANALYTIC SOLUTIONS

When surface tension dominates we have $\beta = 0$, and the cross section of the rivulet is a circular segment. If in addition θ_p is $\pi/2$ or π it is possible to obtain the solution in closed form.

A. The solution for $\beta = 0$, $\theta_p = \pi/2$

The equation (18) can be reduced to Laplace equation setting

$$u = -\frac{1}{2}(t - \zeta_p)^2 + \tilde{u}, \qquad (20)$$

thus obtaining

$$\tilde{u}_{ss} + \tilde{u}_{tt} = 0. \qquad (21)$$

We use polar coordinates r, θ, then $s = r\cos\theta$, $t = r\sin\theta$, the free surface is $r = 1$ ($\eta = \cos\theta$, $\zeta = \sin\theta$, $0 \le \theta \le \pi$) and $d\zeta/d\eta = -\cot\theta$. Then the boundary conditions for $\tilde{u}$ are

$$\tilde{u}_r(1,\theta) = \sin^2\theta, \quad 0 \le \theta \le \pi, \qquad (22)$$

$$\tilde{u}(r,0) = \tilde{u}(r,\pi) = 0. \qquad (23)$$

We first solve (21) in the semicircular domain $0 \le r \le 1$, $0 \le \theta \le \pi$ by separation of variables. The solutions that are regular at $r = 0$ and satisfy (23) can be expressed as

$$\tilde{u} = \sum_{m=1}^{\infty} a_m r^m \sin m\theta. \qquad (24)$$

Using the condition (23) to find the coefficients a_m we finally obtain

$$\tilde{u} = -\frac{8}{\pi}\sum_{n=0}^{\infty} \frac{r^{2n+1}\sin[(2n+1)\theta]}{(2n-1)(2n+1)^2(2n+3)}. \qquad (25)$$

This series can be expressed in terms of the Lerch trascendent function $\Phi\left(r^2 e^{2i\theta}, 2, \frac{1}{2}\right)$. However, in the calculations it is better to use the expression (25). The full solution is obtained using (20) as

$$u = -\frac{1}{2}t^2 + \tilde{u}. \qquad (26)$$

The maximum value of u is

$$u(1,\pi/2) = \frac{1+2C^*}{\pi} - \frac{1}{2} = 0.401432...,$$

where $C^* = 0.915966...$ is Catalan's constant. Contour plots of this solutions can be seen in Fig. 4.

Integrating (26) over the cross section of the rivulet we obtain the volumetric flow as

$$Q = \frac{\rho g \sin\alpha}{\mu}\left(\frac{6 - \pi^2 + 7\varsigma(3)}{4\pi}\right) \cong 0.361663\frac{\rho g \sin\alpha}{\mu}.$$

Here ς is the Riemann's Zeta function.

B. The solution for $\beta = 0$, $\theta_p = \pi$

It can be verified that the solution of (18) that satisfies (19) is

$$u = A - \frac{1}{4}(2 + r^2) + \frac{1}{2}\ln(1 + r^2 + 2r\sin\theta),$$

where the constant A must be chosen to satisfy (19). This solution diverges, as must be expected since for $\beta = 0$, $\theta_p = \pi$ the rivulet touches the supporting plane along a line, so that the viscous drag exerted by the plane vanishes.

VI. NUMERICAL SOLUTIONS

For arbitrary β and θ_p the velocity field must be calculated numerically. We have used a finite element method

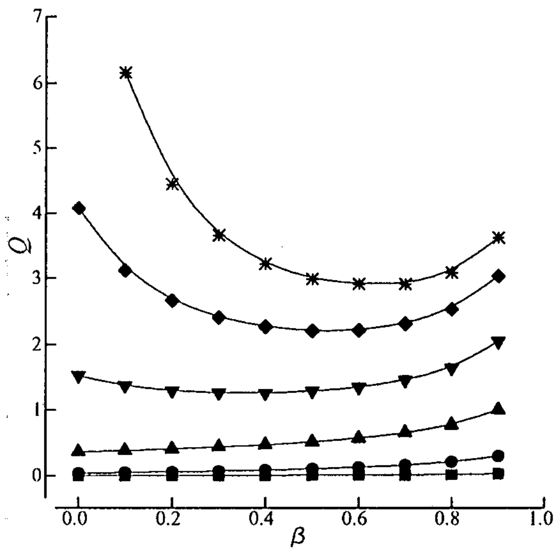

FIG. 5: The volumetric flow Q.

to integrate (18) subject to the boundary conditions (19). The mesh varied from 9000 to 70000 points according to the shape of the contour. We checked the accuracy of the method by comparing the numerical solution for $\beta = 0$, $\theta_p = \pi/2$ with the closed form solution (26). The average difference between the numerical and the exact u was ≈ 0.0015 % and the maximum difference was ≈ 0.05 %.

We calculated solutions for 10 values of β from 0 to 0.9 and 6 values of θ_p from $\pi/6$ to π. In Fig. 4 we show typical contour plots of $u(s,t)$. Notice that the larger velocity gradients occur as expected near the supporting plane and not too close to the surface of the rivulet, where the isovelocity contours are nearly parallel to the plane. On approaching the free surface the isovelocity contours become perpendicular to the surface.

From the numerical solutions it is easy to compute the volumetric flow $Q = \int u \, dS$ (Fig. 5), the cross section of the rivulet, the maximum value of the velocity at the vertex, and the average velocity. In terms of these quantities and of the geometrical properties of the free surface derived in Sec. III any other magnitude of interest (such as the drag coefficient) can be computed.

VII. CONCLUSION

We derived exact solutions of the Navier-Stokes equations that describe the steady flow of rivulets running down an inclined surface. We find that the shape of the free surface is given by an analytic formula obtained by solving the equation that expresses the condition of static equilibrium under the action of gravity and surface tension, independently of the velocity field and on any assumption concerning the rheology of the liquid. The velocity field is then obtained by solving (in general numerically) a Poisson equation in the cross section of the rivulet. The isovelocity contours are always perpendicular to the free surface. Two special analytic solutions are presented.

The present solutions are valid for arbitrary values of the Reynolds number of the flow. However, the issue of their stability remains to be investigated in the future.

Acknowledgments

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