Steady and traveling wave solutions describing the spilling of a power-law liquid film over an incline

Carlos A. Perazzo[1] and Julio Gratton[2,*]

[1] *Universidad Favaloro, Solís 453, 1078, Buenos Aires, Argentina*
[2] *INFIP-CONICET, Dpto. de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Ciudad Universitaria, Pab. I, Buenos Aires, Argentina.*

El flujo lento de películas delgadas de un líquido sobre superficies sólidas es un importante fenómeno en la naturaleza y en procesos industriales. La investigación teórica y numérica en este tema se ha centrado sobre fluidos Newtonianos, a pesar que frecuentemente tanto en situaciones reales como experimentales la reología del líquido involucrado no es Newtoniana. Usando la aproximación de lubricación, obtenemos la familia completa de soluciones estacionarias y ondas viajeras que describen corrientes unidimensionales de un líquido con reología del tipo ley de potencia sobre un plano inclinado bajo la acción de la gravedad y de los esfuerzos viscosos, para valores arbitrarios del exponente reológico.

The slow flow of thin liquid films on solid surfaces is an important phenomenon in nature and in industrial processes. So far the theoretical and numerical research has been focused on Newtonian fluids, notwithstanding that often in the real situations and in the experiments, the rheology of the involved liquid is not Newtonian. Using the lubrication approximation we obtain the full family of steady and traveling wave solutions that describe one-dimensional currents of a power-law non-Newtonian fluid on an inclined plane under the action of gravity and the viscous stresses, for arbitrary values of the rheological exponent.

I. INTRODUCTION

The slow flow of a thin liquid film is an ubiquitous phenomenon in nature as well as in artificial instances. The theory of these currents is usually based on the lubrication aproximation [1], that assumes that the motion is essentially parallel to the supporting plane, so that the pressure is purely hydrostatic, that inertia effects are negligible, and that the length of the current is much larger that is depth. Flows on a horizontal plane have been studied by several authors [2–5]. The equations for the same problem but on a general topography have been also derived in [1]. In all these works it is assumed that the liquid is Newtonian, notwithstanding that often the fluids involved in the real situations are not Newtonian. There are few papers where the non-Newtonian behaviour is considered. The governing equations of slow gravity flows of a power-law fluid on a horizontal plane and on an incline were derived in [6] and [7] respectively, and in [8, 9] the flow of a Bingham fluid on an incline is studied. The viscoplastic Herschel-Bulkley model has been employed to study mud flows down a slope [10].

Here we investigate theoretically the traveling wave solutions describing the flow of a power-law liquid on an incline. We employ the governing equations that describe the evolution of the free surface and the velocity of the fluid under the effect of gravity and viscous stresses, obtained in [7] within the lubrication approximation. In Sec. II we show that there are 17 different kinds of traveling waves, according to the value of the propagation velocity c and of an integration constant j_0 that is related to the difference beetwen c and the averaged velocity of the fluid u. Closed form expressions for the profiles can be found if c, or j_0, or both vanish. Five of them, that we discuss in Sec. III, are steady solutions ($c = 0$). Next, in Sec. IV we present 8 different downslope traveling waves. In Sec. V we investigate 4 upslope traveling waves. Some of the solutions presented in Sections IV and V correspond to peculiar boundary conditions that may be difficult to achieve in the laboratory, but most correspond to simpler conditions, fairly easy to reproduce. In Sec. VI we give the expressions of the 17 solutions for Newtonian liquids, since they can be obtained in terms of known functions and many of them have not been published. Section VII deals with the special cases of flows on horizontal and vertical planes. Section VIII contains the final discussion.

II. BASIC EQUATIONS

We consider a fluid moving on a non horizontal plane, whose inclination is α (Fig. 1). The coordinate z is perpendicular to the plane, and the x coordinate lies in the plane and increases downwards. We shall assume that the y coordinate is ignorable.

We assume the power-law constitutive equation [11]

$$\tau_{ij} = 2AE^{(1-\lambda)/\lambda}\dot{\varepsilon}_{ij},$$

where $\dot{\varepsilon}_{ij}$ is the strain rate tensor, $E = (\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij})^{1/2}$ is its second invariant, and A, λ are positive constants. The power-law rheology is the simplest non-Newtonian model, and with appropriate choices of A and λ according to the strain rates of the problem at hand, it describes

*Electronic address: jgratton@tinfip.lfp.uba.ar

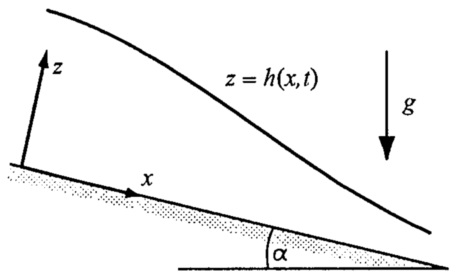

FIG. 1: Geometry of the problem.

reasonably well the behavior of many fluids of practical interest. Shear thinning liquids correspond to $\lambda > 1$, shear thickening fluids to $\lambda < 1$, and the Newtonian rheology to $\lambda = 1$, in which case A is the viscosity.

Let $z = h \equiv h(x,t)$ be the free surface of the current. We assume that the length of the current is much larger than its depth, then the flow is almost parallel to the plane and the lubrication approximation can be used. Then the equation that governs the problem [7] is

$$h_t + k\sigma \left\{ [\sigma(\tan\alpha - h_x)]^\lambda h^{\lambda+2} \right\}_x = 0, \qquad (1)$$

where $k \equiv A^{-\lambda}(\lambda+2)^{-1}2^{(1-\lambda)/2}(\rho g \cos\alpha)^\lambda$ and $\sigma \equiv \mathrm{sgn}(\tan\alpha - h_x)$. In terms of h, the (z averaged) velocity u and the flow j (per unit length in the y direction) of the current are given by

$$u = k\sigma [\sigma(\tan\alpha - h_x)]^\lambda h^{\lambda+1}, \quad j = uh.$$

Note that the direction of the flow is determined by the slope of the free surface with respect to the horizontal.

To find traveling wave solutions we assume that h depends on the single variable $s \equiv x - ct$, where c is a constant. Then Eq. (1) can be integrated to obtain

$$\frac{dh}{ds} = \tan\alpha - \sigma \left[\frac{\sigma(j_0 + ch)}{kh^{\lambda+2}} \right]^{1/\lambda}, \qquad (2)$$

where j_0 is an integration constant. Now u and j can be expressed in terms of h by

$$u = c + \frac{j_0}{h}, \quad j = j_0 + ch.$$

The first of these equations implies that j_0 is the flow as mesured in a reference frame moving with the velocity c. The second equation shows that j_0 is equal to the flow in the points (if they exist) where the thicknes of the current vanishes. These two properties will be useful to interpret the physical meaning of the solutions. We shall see that the number of different types of solutions of (2) and their behaviour is determined by the zeros of the r.h.s. of (2), that in turn depends on c and j_0.

A point s_f where h vanishes corresponds to a front of the current. Referring to Fig. 1, a front may lie to the left of the current (upslope) or to its right (downslope). In the first case $h = 0$ for $s \le s_f$, and in the second for $s \ge s_f$. If $c = 0$ the front is fixed. If $j_0 \neq 0$ there is a non-vanishing flow at the front. This is due to the presence of a sink that is absorbing this flow, as occurs if the supporting plane has a border where the fluid spills over [3]. From (2) it can be seen that the profile near a front is given by $h \propto |s - s_f|^{\lambda/(2\lambda+1)}$ if $j_0 = 0$ or $h \propto |s - s_f|^{\lambda/(2\lambda+2)}$ if $j_0 \neq 0$ (u diverges at the sink).

For $\lambda \neq 1$, the equation (2) must be integrated numerically, except when j_0, or c, or both vanish, in which cases the following implicit formula for $h(s)$ can be given:

$$s\tan\alpha + B = h\left\{1 - {}_2F_1\left[\mu, 1; \mu+1; \sigma\,(h/h_0)^{\frac{1}{\mu}}\right]\right\}, \qquad (3)$$

where ${}_2F_1(a,b;c;z)$ is the hypergeometric function, and B is an integration constant. When $c = 0$ we must use in the previous equation

$$\mu \equiv \frac{\lambda}{\lambda+2}, \quad h_0 = h_j \equiv \left(\frac{\sigma j_0}{k\tan^\lambda\alpha}\right)^{1/(\lambda+2)}, \qquad (4)$$

while when $j_0 = 0$ we must set

$$\mu \equiv \frac{\lambda}{\lambda+1}, \quad h_0 = h_c \equiv \left(\frac{\sigma c}{k\tan^\lambda\alpha}\right)^{1/(\lambda+1)}. \qquad (5)$$

The solutions for $j_0 = 0$ have been discussed in [7]. Notice that ${}_2F_1(\mu, 1; \mu+1; z)$ is complex for $z > 1$, and its imaginary part is $-\pi\mu z^{-\mu}$. Then whenever $z > 1$ we must set $\mathrm{Im}(B) = \pi\mu h_0$ to obtain a real solution.

III. STEADY SOLUTIONS

These solutions arise when $c = 0$. There are two trivial steady solutions, the first is

$$h = s\tan\alpha + \mathrm{const.}\ , \ j_0 = 0, \qquad (6)$$

and describes a fluid resting on an incline. The other is

$$h = h_0 = \mathrm{const.}\ , \ j_0 = kh_0^{(\lambda+2)}\tan^\lambda\alpha, \qquad (7)$$

that represents a uniform flow on an incline. Using (3) and (4) three other steady solutions are obtained. In Fig. 2 we show the five steady solutions. This and the following figures have been drawn for $\lambda = 1.5$. For different values of λ the shape of the profiles looks similar.

To avoid lengthy circumlocutions when describing the currents, we introduce a shorthand notation. We label H (for horizontal) the solution (6) and P (for parallel) the solution (7). Any current having a part that tends to the horizontal will have H in its label. Any solution having a part that tends to be parallel to the supporting plane will have P in its label. When $j_0 \neq 0$ we add to the label a + or − sign according to the sign of j_0. A sink to the right (downslope) will be denoted by r and one to the left (upsolpe) by l. With this notation the solutions

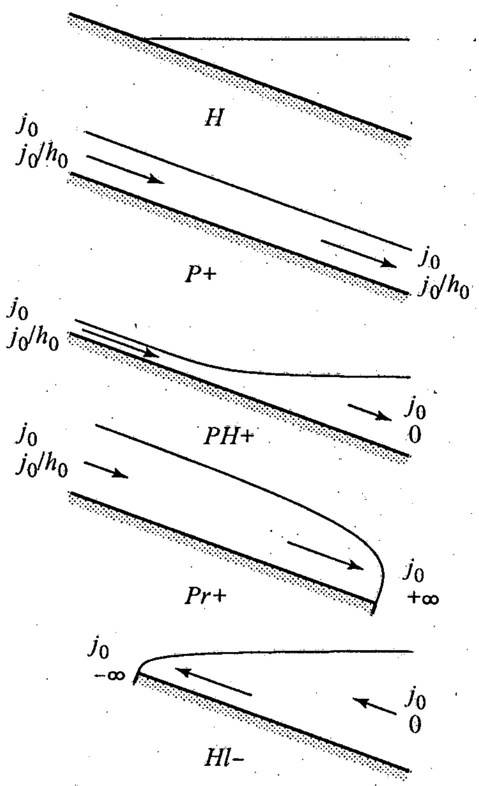

FIG. 2: Steady solutions for $\lambda = 1.5$. In this and the following figures we give the limiting values of j and u to the left and to the right of each profile. A thin arrow indicates the direction of the velocity of the fluid.

shown in Fig. 2 are: H, $P+$, $PH+$, $Pr+$ and $Hl-$. The meaning of the last three solutions is:

• $PH+$: this solution is a steady current flowing into a large and deep pool limited far to the right by a retaining wall of fixed height, so that the excess fluid spills over it. Its profile is obtained from (3) with h_0 and μ given by (4), in which we must set $j_0 > 0$, and $h > h_0$. $PH+$ is the simpler combination of $P+$ and H.

• $Pr+$: it describes a steady current flowing down a half-plane over whose border it spills. Its profile is given by (3) with h_0 and μ given by (4) with $j_0 > 0$ as before, but now $h < h_0$. This solution is a variation of $P+$.

• $Hl-$: this solution occurs when a large and deep pool limited to the left by an inclined wall of fixed height is being filled with a constant flow so that the excess fluid spills over the inclined wall. Its shape is given by (3) with h_0 and μ given by (4) with $j_0 < 0$, and is a variant of H.

IV. WAVES TRAVELING DOWNSLOPE

Figure 3 displays waves traveling downslope. To indicate this fact we add to their labels an arrow pointing down. Also we employ an asterisk to distinguish a solution that has a moving front, when it is not a moving sink. We now describe the most interesting:

• $PH\downarrow$: this solution occurs when a steady current flows down a plane into a large and deep pool that is being drained in such a way that its level lowers at a constant rate, such that $u = c$ (since $j_0 = 0$). This solution is described by equations (3) and (5).

• $P^*\downarrow$: this solution describes the current produced when a source located far upslope has started delivering a constant flow some time in the past.

• $PH\downarrow+$: this solution is similar to $PH\downarrow$, the difference being that now the rate of lowering of the level of the pool is slower than for $PH\downarrow$, so that $u > c$ ($j_0 > 0$).

• $PH\downarrow-$: this solution describes a situation in which the level of the pool lowers faster than in the case of $PH\downarrow$, and then $u < c$ ($j_0 < 0$).

• $PP'\downarrow-$: here P' indicates that this current tends asymptotically to two different values of h (upslope and downslope). This solution is analogous to the kinematical flood wave [12] and describes the current produced when a steady source far upslope increases suddenly its flow to a new constant value. Notice that in this case j_0 must satisfy the condition $j^* < j_0 < 0$ with

$$j^* = -ch_c(\lambda+1)(\lambda+2)^{-\frac{(\lambda+2)}{(\lambda+1)}}.$$

When considering this solution it may be convenient to express c and j_0 in terms of h_{01}, $j_1 = j_0 + ch_{01}$ and h_{02}, $j_2 = j_0 + ch_{02}$, as these quantities are directly related to the behaviour of the source. One finds

$$c = \frac{j_1 - j_2}{h_{01} - h_{02}}, \quad j_0 = -\frac{h_{02}j_1 - h_{01}j_2}{h_{01} - h_{02}},$$

with $j_i = kh_{0i}^{\lambda+2}\tan^\lambda\alpha$, $i = 1, 2$. It can be verified that the condition $j^* < j_0 < 0$ does not restrict the ratio h_{01}/h_{02}, that can take any value larger than unity.

We shall not discuss the three remaining solutions ($Pr\downarrow+$, $PHl\downarrow-$, $Pl\downarrow-$) as they have a moving sink, a feature that does not occur in situations of practical interest.

V. WAVES TRAVELING UPSLOPE

Two of these solutions (Fig. 4) are interesting:

• $H^*\uparrow$: it describes a large pool with an inclined side, that is being filled from far right in such a way that the level increases at a constant rate. The corresponding formula for h is given by (3) and (5) since $j_0 = 0$.

• $PH\uparrow+$: this solution corresponds to the same situation of $H^*\uparrow$, but in addition a constant flow is feeding the pool from upslope. In this current there is a point in which the slope of the free surface is horizontal. To the left of this point the fluid moves downslope, while to the rigth of it the liquid moves upslope.

The two remaining solutions ($Pr\uparrow+$ and $Hl\uparrow-$) have a moving sink, and we shall not discuss them.

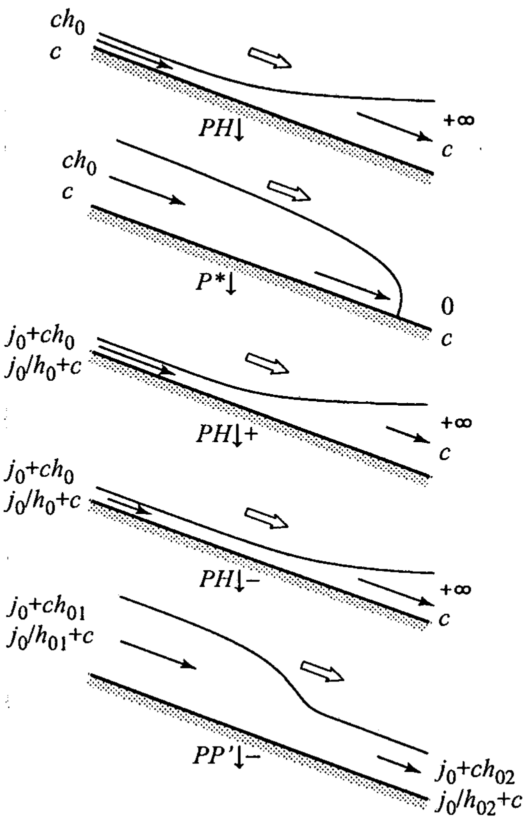

FIG. 3: Waves traveling downslope ($\lambda = 1.5$). In this figure and the next a hollow wide arrow indicates the sense of motion of the wave.

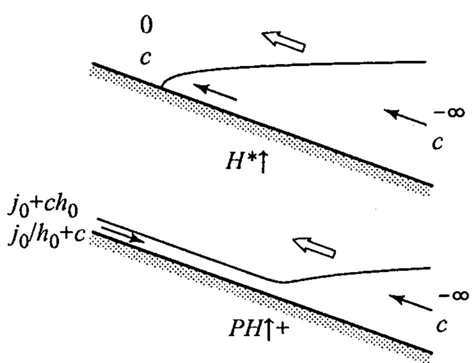

FIG. 4: Waves traveling upslope ($\lambda = 1.5$).

VI. THE NEWTONIAN CASE

When the liquid is Newtonian ($\lambda = 1$) all the 17 solutions can be obtained in closed form. Since they have not been published except for the special case $\alpha = 0$ (see [3]), we shall give the corresponding expressions that can be obtained from the general formula

$$s \tan\alpha + B = h - \sum_{h_r} \frac{j_0 \ln(h-h_r) + ch_r \ln(h-h_r)}{c - 3k \tan\alpha h_r^2}, \tag{8}$$

where the sum is over the three roots h_r of $j_0 + ch_r - k \tan\alpha h_r^3 = 0$. When $c = 0$ equation (8) takes the simpler form

$$\begin{aligned} s\tan\alpha + B \;&=\; h - \frac{\sigma h_j}{\sqrt{3}} \arctan\left(\frac{2h+\sigma h_j}{\sqrt{3}\sigma h_j}\right) \\ &+ \frac{\sigma h_j}{6} \ln\left[\frac{(h-\sigma h_j)^2}{h^2+\sigma h h_j + h_j^2}\right]. \end{aligned}$$

When $j_0 = 0$ equation (8) yields

$$s\tan\alpha + B = \begin{cases} h - \frac{h_c}{4} \ln\left[\left(\frac{h+h_c}{h-h_c}\right)^2\right] & , \; c > 0 \\ h - h_c \arctan\left(\frac{h}{h_c}\right) & , \; c < 0 \end{cases}.$$

VII. THE CASES $\alpha = 0$ AND $\alpha = \pi/2$

It is convenient to discuss separately the case when the supporting plane is horizontal as the basic solutions H and P can not be distinguished. This leads to a reduction in the number of kinds of currents and in addition it is posible to obtain expressions in terms of known functions, that are the generalization to non-Newtonian liquids of those obtained in [3].

There are two steady solutions: the trivial one

$$h = \text{const.},$$

corresponding to a uniform layer of fluid, and

$$h = \left(\frac{\sigma j_0}{k}\right)^{\frac{1}{2\lambda+2}} \left[\frac{2\lambda+2}{\lambda\sigma}(s_0 - x)\right]^{\frac{\lambda}{2\lambda+2}},$$

that represents the situation in which the fluid is supported by a half plane ($x < s_0$ if $\sigma = 1$ or $x > s_0$ if $\sigma = -1$) so that at $x = s_0$ the fluid provided by a source located at infinity spills over the border.

The solution

$$h = \left(\frac{\sigma c}{k}\right)^{\frac{1}{2\lambda+1}} \left[\frac{2\lambda+1}{\lambda\sigma}(s_0 - s)\right]^{\frac{\lambda}{2\lambda+1}}$$

represents a current with a front that advances with constant speed c on an infinite plane. It describes the flow produced by a plane piston (or a spatula) that advances with uniform speed, pushing a constant volume of fluid. This solution was already obtained in [6].

The solution ($\lambda_1 \equiv 1 - \lambda^{-1}$, $\lambda_2 \equiv 2 + 2\lambda^{-1}$)

$$s_0 - s \;=\; \sigma \left|\frac{j_0}{c}\right|^{\lambda_2} \mathrm{Re}\left[\left(-\frac{k}{|j_0|}\right)^{\frac{1}{\lambda}} B_{h|c/j_0|}(\lambda_2, \lambda_1)\right],$$

$$h\left|\frac{c}{j_0}\right| > 1,$$

where $B_z(a,b)$ denotes the incomplete beta function, describes for $\sigma = 1$ ($\sigma = -1$) a current in which the thickness of the fluid tends to a constant value $h_0 = |j_0/c|$ far to the right (left), and increases as $|s|^{1/(2\lambda+1)}$ far to the left (right). There is no sharply defined front, but the point where $h = 2h_0$ can be conventionally taken as definig the position of the "front". The average velocity of the fluid tends to zero far ahead this "front", and approaches the constant value c far behind. This solution is the asymptotycs of the current produced by a piston that pushes a layer of fluid of uniform thickness h_0, when a sufficiently long time has elapsed from the beginning of the motion, and the perturbation has advanced to a very great distance from the piston.

The two remaining solutions are

$$s_0 - s = \sigma \left|\frac{j_0}{c}\right|^{\lambda_2} \mathrm{Re}\left[\left(-\frac{k}{|j_0|}\right)^{\frac{1}{\lambda}} B_{h|c/j_0|}(\lambda_2, \lambda_1)\right], \quad h\left|\frac{c}{j_0}\right| < 1,$$

and

$$s_0 - s = \sigma \left|\frac{j_0}{c}\right|^{\lambda_2} \left(\frac{k}{|j_0|}\right)^{\frac{1}{\lambda}} e^{-\frac{2\pi i}{\lambda}} B_{-h|c/j_0|}(\lambda_2, \lambda_1).$$

Both represent currents with moving sinks.

In the limit $\alpha = \pi/2$ only the solutions H and $P+$ survive, together with combinations in which the horizontal and vertical parts of the free surface join at rigth angles.

VIII. DISCUSSION AND CONCLUSIONS

According to the above descriptions, we find that if one disregards the five solutions that have moving sinks ($Pr\downarrow+$, $PHl\downarrow-$, $Pl\downarrow$, $Pr\uparrow+$ and $Hl\uparrow-$), the remaining 12 solutions can be separated into two groups.

The first group consists of flows into a very large pool bounded on one side by an inclined plane (either with or without a border to the left). This comprises H, $Hl-$, $PH+$, $PH\downarrow$, $PH\downarrow+$, $PH\downarrow-$, $PH\uparrow+$, and $H^*\uparrow$. All the currents of this group can be produced by a device that empties or fills the pool in such a way that the level of the liquid lowers or rises at a constant rate, plus an appropiate source far upslope on the inclined plane.

The second group consists of solutions representing flows produced by a source far upslope of an inclined plane (with or without a border to the right). It comprises $P+$, $Pr+$, $P^*\downarrow$ and $PP'\downarrow-$.

As one changes the rheological parameter λ the general shape of the different solutions looks similar. However it must be kept in mind that the properties of the solutions are sensitive to λ, that enters in a non trivial way into the basic equations (1)-(2). It can also be noticed that several solutions we have described have their counterparts in the case of a Bingham fluid as can be observed in [8].

The present theory does not include surface tension effects, which implies that the appropriate Bond number must be large. Surface tension effects will be more relevant where the curvature of the free surface is large, as happens near a front. But precisely there, the lubrication approximation breaks down, so that a correct description of the current where the curvature is large requires a different, more complex approach that is beyond the scope of this paper. This problem is also present in the Newtonian case, but does not invalidate the remaining parts of the solutions nor the spreading relations derived from the theory (see the discussions in [6, 7]). We do not see any reason why the same should not hold true for non-Newtonian fluids.

This rich family of solutions can be of interest to test the accuracy of numerical simulation codes in a variety of different situations. It should be interesting to compare them with experimental results, but to our knowledge most of them have not yet been investigated in the laboratory.

Acknowledgments

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