# Waiting-time solutions of nonlinear diffusion equations: Corner layer evolution and bounds for the waiting time


**C. A. Perazzo[1], J. Gratton[2-3], C. L. M. Vigo[4]**

*1 Universidad Favaloro*
*2 INFIP-CONICET, Facultad de Ciencias Exactas y Naturales-Univ. de Buenos Aires*
*3 Investigador del CONICET*
*4 Becario Postdoctoral del CONICET*
jgratton@tinfip.lfp.uba.ar



Mediante un código numérico resolvemos la ecuación de difusión no lineal para perfiles iniciales del tipo de ley de potencias que llevan a soluciones con tiempo de espera. De esta forma estudiamos en detalle la formación y evolución del *corner layer* y determinamos el tiempo de espera y la velocidad de arranque del frente en función del exponente de no linealidad y el exponente que caracteriza el perfil inicial. Mostramos que a partir de los valores del tiempo de espera así determinados es posible obtener cotas inferiores y superiores del tiempo de espera para condiciones iniciales más generales. Dichas cotas son más restrictivas que las cotas teóricas hasta ahora conocidas.

We solve numerically the nonlinear diffusion equation for initial profiles of the power law type that yield waiting time solutions. We study in detail the formation and evolution of the corner layer and we find the waiting time and the start-up velocity as functions of the nonlinearity exponent and of the exponent characterizing the initial profile. We show that from the waiting-times thus obtained it is possible to derive lower and upper bounds of the waiting-time for more general initial conditions. These bounds are more stringent than the known theoretical ones.


## I. Introduction

Many phenomena such as flows in porous media, viscous-gravity currents, etc. are described by nonlinear diffusion equations of the type

$$\eta_t = \tfrac{1}{m}(\eta_x)^2 + \eta\eta_{xx} \qquad (1)$$

where $\eta = \eta(x,t) \geq 0$ and $m > 0$. Under conditions that will be specified below, its solutions display the waiting-time behavior, that consists of an initial time interval $t_w$ in which the $\eta$ profile changes, but its front remains motionless. More precisely, if initially ($t = 0$) $\eta > 0$ for $x > x_f$ and $\eta = 0$ for $x \leq x_f$, then also $\eta > 0$ for $x > x_f$ and $\eta = 0$ for $x \leq x_f$ for $0 \leq t \leq t_w$. We shall be concerned with solutions of (1) defined for $x \leq 1$ (with some adequate boundary condition at $x = 1$).

Given the initial profile $\eta(x,0) = g(x)$, the theory predicts whether the solution will have a non vanishing waiting-time, but excepting very few special cases it does not give the value of $t_w$. In the remaining cases, it only provides lower and upper bounds of $t_w$, that frequently are poor estimators of the exact value.

It has been proved[1,2]. that if $g(x) = 0$ for $x < 0$ and $g(x\to 0+) = A\,x^{2\alpha}$, with $\alpha \geq 1$ the solution is of the waiting-time kind; otherwise the front starts moving immediately. Furthermore, if $\alpha > 1$, during the waiting time the profile develops a narrow moving region, called *corner layer*, in which $\eta_{xx}$ has a narrow peak. As the corner layer moves towards the waiting front it becomes stronger (*i.e.*, the peak width decreases and its height increases), and on arriving at the front it becomes a corner shock (a discontinuity of $\eta_x$), and the waiting-time interval ends. In these cases the motion of the corner layer determines the value of waiting time $t_w$ and the start-up velocity of the front. On the contrary, when $\alpha = 1$ no corner layer develops during the waiting stage, but a corner shock is formed at the front position precisely at start-up.

In addition, Aronson *et al*[3] proved that, if

$$g(x) = Ax^2 + O(x^2) \text{ as } x \to 0$$

and (2)

$$g(x) \leq Bx^2 \quad \text{for all } x \geq 0$$

then

$$t_B = \frac{m}{2(m+2)B} \leq t_w \leq \frac{m}{2(m+2)A} = t_A \qquad (3)$$

When $A = B$ these bounds coincide, and (3) gives the exact value of $t_w$.

A useful upper bound of $t_w$ (but not the exact value) can be derived by means of the Shifting Comparison Principle of Vázquez[4]. Introducing $h = \eta^{1/m}$ and $G = g^{1/m}$, this Principle can be stated in our case as follows: if for every $x \leq 1$ we have

$$M_1(x,0) = \int_{-\infty}^{x} G_1(x)\,dx \leq \int_{-\infty}^{x} G_2(x)\,dx = M_2(x,0) \qquad (4)$$

then for every $t > 0$ one has

$$M_1(x,t) = \int_{-\infty}^{x} h_1(x,t)\,dx \leq \int_{-\infty}^{x} h_2(x,t)\,dx = M_2(x,t) \qquad (5)$$

*i. e.*, if the "mass" of $\eta_1^{1/m}$ is initially shifted to the right with respect to that of $\eta_2^{1/m}$, this situation is preserved for every later time. According to this principle, if $g_{1,2}(x \leq 0) = 0$ and $\eta_2$ is a waiting-time solution, then $\eta_1$ is also a waiting-time solution with $t_{w1} \geq t_{w2}$, a result that shall be useful later. To obtain the upper bound of Vázquez, let us consider a waiting-time solution $\eta_2$ with $M_2(1,0) = 1$ whose waiting time is $t_{w2}$. Now we take $G_1 = 2\delta(x-1)$ that is an initial condition for which (1) has a known analytical solution[4], according to which $M_1(0,t) = 0$ for $t \le t_\infty$ given by

$$t_\infty(m) = \frac{m}{2m+4}\left[\frac{\sqrt{\pi}\,\Gamma\left(1+\frac{1}{m}\right)}{2\Gamma\left(\frac{3}{2}+\frac{1}{m}\right)}\right]^m \tag{6}$$

Then according to the Comparison Principle

$$t_{w2} \le t_\infty. \tag{7}$$

that is the desired upper bound. If $M_2(1,0) = A$ we must take $G_1 = 2A\delta(x-1)$, and correspondingly the bound is then $t_\infty/A$.

In another paper[5] we obtained numerically the exact values of $t_w$ for initial conditions of the type

$$g(x) = \begin{cases} Kx^{2\alpha} & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x \le 0 \end{cases} \tag{8}$$

for $\frac{1}{2} \le m \le 9$ and $1 < \alpha \le 12$. We used $K = (1+2\alpha/m)^m$ (so that $M(1,0) = 1$), and the boundary condition $\eta_x(1,t) = 0$. We defined the initial time as $-t_w$, so that the waiting stage ends at $t = 0$.

Here we investigate the formation and evolution of the corner layer, and we show how to employ our exact values of $t_w$ to obtain bounds more stringent than the theoretical ones, for initial conditions more general than (8).

## II. Corner Layer motion and development

To describe quantitatively the evolution of the corner layer it is convenient to introduce some definitions that shall be given when needed.

To investigate the motion of the corner layer we define its position $x_c(t)$ as the point where $\eta_{xx}$ is maximum. In Fig. 1 we show the corner layer velocity $v_c(t)$ as a function of $t/t_w$.

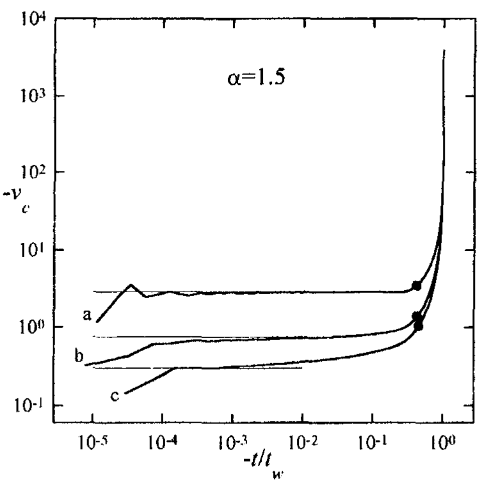


*Figure 1. Velocity of the corner layer for α=1.5 (a: m=1; b: m=4; c: m=9). The point in each curve indicate the time of formation of corner layer(see below), and the horizontal lines indicate the velocity at start-up (as determined from the motion of the front).*

It can be noticed that near the end of the waiting stage, $v_c$ tends to a constant value $c$. This value coincides with the velocity of the front at the start-up, that we can determine independently of the motion of the corner layer. It can be appreciated that $c$ is approached more rapidly for small $m$. A similar behavior can be observed for all $\alpha$.

To study the development of the corner layer we define its width $\Delta x_c(t)$ as the width of the $\eta_{xx}$ peak at half maximum, and its strength as $\Psi(t) = \eta_{xx}(x_c)/\Delta x_c(t)$. To compare the evolution of the corner layer for different $m$ and $\alpha$ we define the normalized strength as $\Pi(t) = \Psi(t)/\Psi(-t_w)$.

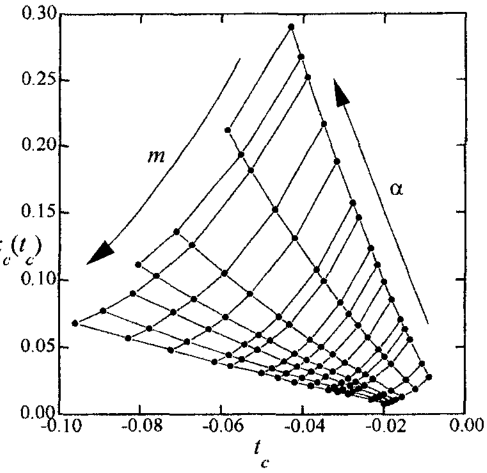


*Figure 2. Time and place of formation of the corner layer for different m and α.*

The $\eta_{xx}$ peak develops gradually; then, strictly speaking, a time of formation of the corner layer cannot be defined. However, to quantify the $\eta_{xx}$ peak development, we define a "time of formation" $t_c$ as the time when $\Pi$ exceeds a threshold that we arbitrarily take as 100, and the "place of formation" as $x_c(t_c)$. Notice that these definitions depend on the determination of $\Delta x_c(t)$, that cannot be achieved if the peak comprises very few grid points. For this reason we accepted only the cases for which $\Delta x_c(t_c)$ includes at least ten grid points.

In Fig. 2 we present $t_c$ vs. $x_c(t_c)$. It can be appreciated that, for fixed $\alpha$, as $m$ increases the corner layer develops closer to the front and earlier during the waiting stage. On the other hand, as $\alpha$ decreases for fixed $m$, the corner develops closer to the front, but later during the waiting stage.

## III. The waiting time and its bounds

For convenience we show in Fig. 3 the numerical $t_w$ obtained in[5] for the initial conditions (8).

For each $m$, $t_w$ is an increasing function of $\alpha$, and tends to $t_\infty$ for $\alpha \to \infty$. On the other hand, for $\alpha = 1$, the waiting time can be obtained from Eqs. (2) and (3), and its value is

$$t_1 = \frac{1}{2}\left(\frac{m}{m+2}\right)^{m+1} \tag{9}$$

For the initial conditions (8), Eqs. (2) and (3) do not give a finite upper bound (except for $\alpha = 1$), and the lower bound is a poor estimate of $t_w$ (except for $\alpha \approx 1$).

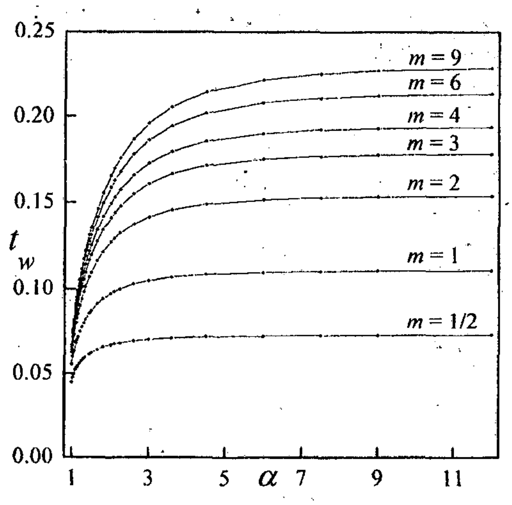


*Figure 3. Waiting time as function on α for all m.*

Our investigation is limited to a particular class of initial profiles, and the reader may wonder what happens in problems involving initial conditions more general than (8). In this connection it must be pointed out that the results of Fig. 3 may be used in combination with the previously mentioned Shifting Comparison Principle, to find lower and upper bounds on $t_w$. Indeed, if we can find $\alpha_1$ and $\alpha_2$ such that for every $x \in (-\infty, 1]$

$$\int_{-\infty}^{x} G_{\alpha 1}(x)\,dx \le \int_{-\infty}^{x} G(x)\,dx \le \int_{-\infty}^{x} G_{\alpha 2}(x)\,dx \qquad (10)$$

where $g_{\alpha 1}(x)$ and $g_{\alpha 2}(x)$ are of the form (8), then $t_{w1} \ge t_w \ge t_{w2}$. Clearly, as long as the bound (6) is more stringent than the upper bound (3), we can achieve with adequate choices of $g_{\alpha 1}(x)$ and $g_{\alpha 2}(x)$ a better upper bound, since our set of comparison functions is broader than that used by Vázquez (that corresponds to the special choice $\alpha_2 = \infty$ in (10)). In the remaining cases we must compare the bounds derived by our method with those given by (3) to verify if they are actually an improvement.

As an example, we take the initial conditions

$$g(x) = \begin{cases} (1-\theta)sen^2\left(\frac{\pi}{2}x\right) + \theta sen^4\left(\frac{\pi}{2}x\right) & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x \le 0 \end{cases} \qquad (11)$$

where $0 \le \theta \le 1$. The initial profile (11) has been studied to test the theoretical bounds. From Eqs. (2) and (3) it can be found that if $0 \le \theta \le 1/4$ then $t_w = 2m[\pi^2(m+2)(1-\theta)]^{-1}$, but if $1/4 < \theta \le 1$, $t_B \le t_w \le 2m[\pi^2(m+2)(1-\theta)]^{-1}$ (notice that $t_B$ must be found numerically and that the upper bound diverges as $\theta \to 1$).

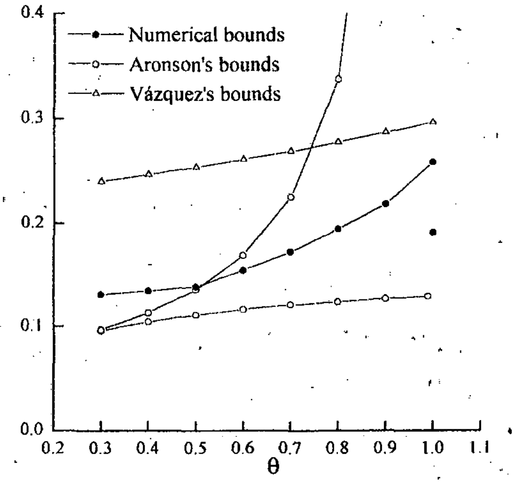


*Figure 4. Theoretical and numerical bounds of $t_w$ for the initial condition (11).*

For the profile (11), except for the special case $\theta = 1$, we cannot improve the lower bound. To test our upper bounds we considered the case $m = 1$. The results are shown in Fig. 4. It can be seen that the upper bound we derive is more stringent than (3) for $\theta \gtrsim 0.55$ and is always better than (6) as anticipated. For the special case $\theta = 1$ we find also a lower bound that is roughly 50% higher than that derived from (3).

In conclusion, we find that our numerical results can be used to obtain useful bounds for the waiting time, in instances when is not known exactly.

This work was supported by grants of the University of Buenos Aires (Project TW16) and of CONICET (PIP4521).

## References

[1] W.L. Kath and D.S. Cohen, Stud. Appl. Math. **67**, 79 (1982)

[2] J. L. Vázquez,Trans. Amer. Math. Soc. **285**, 717 (1984)

[3] D.G. Aronson, L.A. Caffarelli and S. Kamin, SIAM J. Math. Anal. **14**, 639 (1983)

[4] J. L. Vázquez,Trans. Amer. Math. Soc. **277**, 507 (1983)

[5] C.A. Perazzo, C.L.M. Vigo, and J. Gratton, Anales AFA **9**, 99-103 (1997)